Constant-Weight and Constant-Charge Binary Run-Length Limited Codes

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Abstract

Constant-weight and constant-charge binary sequences with constrained run length of zeros are introduced. For these sequences, the weight and the charge distribution are found. Then, recurrent and direct formulas for calculating the number of these sequences are obtained. With considering these numbers of constant-weight and constant-charge RLL sequences as coefficients of convergent power series, generating functions are derived. The fact, that generating function for enumerating constant-charge RLL sequences does not have a closed form, is proved. Implementation of encoding and decoding procedures using Cover's enumerative scheme is shown. On the base of obtained results, some examples, such as enumeration of running-digital-sum (RDS) constrained RLL sequences or peak-shifts control capability are also provided.

I. Introduction

The sequences with constrained run length of zeros are known in literature as dk sequences. In these sequences, single ones are separated by at least d, but not more then k zeros. A dkr sequence is a dk sequence, ending in a run of not more then r trailing zeros. A $dklr$ sequence is a dkr sequence, beginning with a run of not more then l leading zeros. These sequences, their properties, and applications are described in [1] in detail or, briefly, in comprehensive overview paper [2].

A general enumerative scheme for encoding and decoding binary sequences has been presented by Cover [3]. We use this technique for determining the number of constrained sequences.

Let $\{0,1\}^n$ be the set of all binary sequences of length n and let $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ denote a generic element of this set. Let $\mathscr{S}(n) = \{\boldsymbol{x} \in \{0,1\}^n \mid \text{satisfies the } d,k,r \text{ constraints}\}$ and let $\hat{\mathscr{S}}(n) = \{\boldsymbol{x} \in \{0,1\}^n \mid \text{satisfies the } d,k,l,r \text{ constraints}\}$.

Using Cover's method, cardinality of $\hat{\mathscr{S}}(n)$ can be computed as shown in [4]. To do this, the number of dkr sequences, which begin with one, is calculated as

$$|\mathscr{S}(n)| = \sum_{j=d+1}^{k+1} |\mathscr{S}(n-j)|, \quad n > d+k.$$

Then the number of $dklr$ sequences is calculated as

$$|\hat{\mathscr{S}}(n)| = \sum_{j=0}^{\min(n,l)} |\mathscr{S}(n-j)|.$$

By $\nu = \sum_{j=1}^{n} x_j$ denote the weight of the sequence $\boldsymbol{x}$. The number of unconstrained constant-weight sequences may be simply obtained as $\binom{n}{\nu}$, see [5]. Methods for calculating the number of constant-weight dkr sequences is given by [6], [7].

Under NRZI encoding [2] we understand mapping the source sequence $\boldsymbol{x}$ to bipolar sequence $\boldsymbol{z}$, $\boldsymbol{z} \in \{-1,1\}^n$ such that

$$z_j = \begin{cases} z_{j-1}, & x_j = 0, \\ -z_{j-1}, & x_j = 1, \end{cases}$$
$$z_0 = 1.$$

By $\sigma = \sum_{j=1}^{n} z_j$ denote the digital sum or charge of the sequence $\boldsymbol{z}$. Observe that $\sigma \in [-n, n]$, where σ admits even values whenever n is even and odd values whenever n is odd. For example, Table I shows sequences $\boldsymbol{x}$ and $\boldsymbol{z}$, theirs ν and σ respectively.

Our central goal will be to determine the number of constant-weight and constant-charge run length limited sequences as well as an exact estimation of this number. We also intend to show that using Cover's enumerative method provides us with necessary values for error control. Namely, these values follow from weight distribution, run-length distribution, and charge distribution, which may be obtained by Cover's technique.

Although we will consider constant-weight and constant-charge RLL sequences together, our main efforts will be focused on constant-charge RLL sequences. Some results concerning the constant-weight RLL sequences are known with the contributions

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TABLE I
ALL LEXICOGRAPHICALLY ORDERED $dklr$
SEQUENCES OF LENGTH $n=8$, WITH
CONSTRAINTS $d=2$, $k=4$, $l=1$, $r=3$.

N[a]	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	ν	
	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8		σ
0	0	1	0	0	0	0	1	0	2	
	1	-1	-1	-1	-1	-1	1	1		-2
1	0	1	0	0	0	1	0	0	2	
	1	-1	-1	-1	-1	1	1	1		0
2	0	1	0	0	1	0	0	0	2	
	1	-1	-1	-1	1	1	1	1		2
3	0	1	0	0	1	0	0	1	3	
	1	-1	-1	-1	1	1	1	-1		0
4	1	0	0	0	0	1	0	0	2	
	-1	-1	-1	-1	-1	1	1	1		-2
5	1	0	0	0	1	0	0	0	2	
	-1	-1	-1	-1	1	1	1	1		0
6	1	0	0	0	1	0	0	1	3	
	-1	-1	-1	-1	1	1	1	-1		-2
7	1	0	0	1	0	0	0	1	3	
	-1	-1	-1	1	1	1	1	-1		0
8	1	0	0	1	0	0	1	0	3	
	-1	-1	-1	1	1	1	-1	-1		-2

[a] By N we denote the lexicographic index of sequence.

coming from Lee [8], Forsberg and Blake [9], Ytrehus [6]. We will cite some of their results for the sake of generalization and in comparison with the similar results for constant-charge codes.

Run-length distribution does not considered in this paper, although the length of each run is accounted in Cover's enumerative technique. The problems of defining this distribution and accompanying problems are studied in source coding. The reader can find these materials in various publications since well-known Huffman's paper [10].

The rest of this paper will be organized as follows. First, in Section II, we derive recursion relations for determining the number of the sequences. Further, in Section III, we obtain direct formulas for the same. Next, in Section IV, we consider the numbers of our sequences as coefficients of formal power series and derive generating functions. We also prove that generating function for enumerating constant-charge RLL sequences does not have a closed form. Then, in Section V, we give an enumerative algorithm for encoding and decoding these sequences. Some remarks and application notes are presented in Section VI. Finally, conclusions are drawn in Section VII.

II. THE NUMBER OF SEQUENCES

Consider run-length constrained binary sequences of length n and weight ν. Let A_n^ν be the number of these sequences which begin with one. Let $\hat{A}_n^\nu$ be the number of these sequences which begin with a leading run of zeros.

Since an internal run of zeros succeeds a leading run of zeros then the leading constraint l does not affect A_n^ν. For convenience, below under A_n^ν we consider $A_n^\nu(d,k,r)$ and under $\hat{A}_n^\nu$ we similarly consider $\hat{A}_n^\nu(d,k,l,r)$.

Suppose that a unique sequence of zero length and zero weight exists. Let it be a sequence which begins with one. Then

$$A_0^0 = 1.$$

Proposition 1. *The numbers A_n^ν and $\hat{A}_n^\nu$ can be obtained as:*
If $\nu = 1$ and the sequences begin with one, then

$$A_n^1 = \begin{cases} 1, & 1 \le n \le r+1, \\ 0, & \textit{otherwise}. \end{cases} \tag{1}$$

If $\nu > 1$ and the sequences begin with one, then

$$A_n^\nu = \begin{cases} 0, & n < d+1, \\ \sum_{j=d+1}^{\min(n,k+1)} A_{n-j}^{\nu-1}, & d+1 \le n. \end{cases} \tag{2}$$

If $\nu = 0$ and a leading series of zeros is running, then

$$\hat{A}_n^0 = \begin{cases} 1, & n \le \min(l,r), \\ 0, & \textit{otherwise}. \end{cases} \tag{3}$$

If $\nu > 0$ and a leading series of zeros is running, then

$$\hat{A}_n^\nu(l,r) = \sum_{j=0}^{\min(n,l)} A_{n-j}^\nu. \tag{4}$$

TABLE II

AN EXAMPLE OF CHARGE CHANGING

(a) After a leading one.

m				x_1	x_2	x_3	x_4	x_5	σ_m
5				1	0	0	1	0	
				-1	-1	-1	1	1	-1
n	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	σ_n
8	1	0	0	1	0	0	1	0	
	-1	-1	-1	1	1	1	-1	-1	-2

(b) After leading zeros.

m				x_1	x_2	x_3	x_4	x_5	σ_m
5				1	0	0	1	0	
				-1	-1	-1	1	1	-1
n	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	σ_n
8	0	0	0	1	0	0	1	0	
	1	1	1	-1	-1	-1	1	1	2

Here $d \geq 0$, $k \geq d$, $l \geq 0$, $r \geq 0$. *Proof:* In the case of $\nu = 1$, there is only a trailing run of zeros in the sequence. It gives us the only allowed sequence which length lies in the interval $[1, r+1]$. Therefore, (1) is evident. In the case of $\nu > 1$, according to Cover's enumerative method [3], we build the recursion by the following way. Let us consider a possible run of zeros, which follows the leading one, as a prefix for the following subsequences beginning also with one. Assuming the length of the prefix grows from $d+1$ to $\min(n, k+1)$ and weight of this prefix equals one, we obtain the number of these subsequences evidently equal $A_{n-j}^{\nu-1}$ for each allowed prefix and the total number is expressed by (2).

The other case is when a leading series of zeros is running. In the case of zero weight, the leading run of zeros is the trailing one. This also gives us the only allowed sequence which length lies in the interval $[0, \min(l, r)]$. Hence, (3) is also evident. In the case of nonzero weight, there exist only zero weight prefixes which length lies in the interval $[0, \min(n, l)]$. Subsequences beginning with one follow this prefixes, thus, the total number of sequences is expressed by (4).

Consider bipolar run-length constrained sequences of length n and charge σ. We can do this in terms of source sequence $\boldsymbol{x}$. This allows us to obtain results similar to Proposition 1. In this case $\sigma = \sum_{j=1}^{n} (-1)^{\nu_j}$, where $\nu_j = \sum_{i=1}^{j} x_i$. Let C_n^σ be the number of these sequences, which begin with one. Let $\hat{C}_n^\sigma$ be the number of these sequences, which begin with a leading run of zeros.

Since an internal run of zeros succeeds a leading run of zeros then the leading constraint l does not affect C_n^σ. For convenience, below under C_n^σ we consider $C_n^\sigma(d,k,r)$ and under $\hat{C}_n^\sigma$ we similarly consider $\hat{C}_n^\sigma(d,k,l,r)$.

Proposition 2. *The numbers C_n^σ and $\hat{C}_n^\sigma$ can be obtained as:*
If $\sigma = -n$ and the sequences begin with one, then

$$C_n^{-n} = \begin{cases} 1, & n \leq r+1, \\ 0, & \textit{otherwise}. \end{cases} \tag{5}$$

If $\sigma \neq -n$ and the sequences begin with one, then

$$C_n^\sigma = \begin{cases} 0, & n < d+1, \\ \sum_{j=d+1}^{\min(n,k+1)} C_{n-j}^{-\sigma-j}, & d+1 \leq n. \end{cases} \tag{6}$$

If $\sigma = n$ and a leading series is running, then

$$\hat{C}_n^n = \begin{cases} 1, & n \leq \min(l, r), \\ 0, & \textit{otherwise}. \end{cases}$$

If $\sigma \neq n$ and a leading series is running, then

$$\hat{C}_n^\sigma = \sum_{j=0}^{\min(n,l)} C_{n-j}^{\sigma-j}. \tag{7}$$

Here $d \geq 0$, $k \geq d$, $l \geq 0$, $r \geq 0$. *Proof:* Is similar to Proposition 1, except the differences between the weight and charge. Namely, if the sequences of length n begin with one, then the charge σ_n can be obtained as

$$\sigma_n = -n + m - \sigma_m,$$

where m and σ_m is the length and charge of following subsequence that also begins with one. Since $m = n - j$, therefore, the number of subsequences in (6) must be $C_{n-j}^{-\sigma-j}$.

If the sequences of length n begin with zero, then the charge σ_n can be obtained as

$$\sigma_n = n - m + \sigma_m.$$

Thus, the number of subsequences in (7) must be $C_{n-j}^{\sigma-j}$. For example, the charge changing is shown in Table II.

TABLE III

AN EXAMPLE OF WEIGHT AND CHARGE DISTRIBUTION

(a) For sequences beginning with one.
Constraints: $d = 2,\ k = 4,\ r = 3.$

	A_n^ν									C_n^σ																
ν,σ / n	0	1	2	3	4	5	6	7	8	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0	1																	1								
1	0	1															1		0							
2	0	1	0													1		0		0						
3	0	1	0	0											1		0		0		0					
4	0	1	1	0	0									1		1		0		0		0				
5	0	0	2	0	0	0							0		1		1		0		0		0			
6	0	0	3	0	0	0	0					0		1		1		1		0		0		0		
7	0	0	3	1	0	0	0	0			0		0		1		2		1		0		0		0	
8	0	0	2	3	0	0	0	0	0	0		0		0		3		2		0		0		0		0

(b) For sequences beginning with a leading run of zeros.
Constraints: $d = 2,\ k = 4,\ l = 1,\ r = 3.$

	$\hat{A}_n^\nu$									$\hat{C}_n^\sigma$																
ν,σ / n	0	1	2	3	4	5	6	7	8	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
0	1																	1								
1	1	1															1		1							
2	0	2	0													1		1		0						
3	0	2	0	0											1		1		0		0					
4	0	2	1	0	0									1		2		0		0		0				
5	0	1	3	0	0	0							0		2		2		0		0		0			
6	0	0	5	0	0	0	0					0		1		2		2		0		0		0		
7	0	0	6	1	0	0	0	0			0		0		2		3		2		0		0		0	
8	0	0	5	4	0	0	0	0	0	0		0		0		4		4		1		0		0		0

Using relations derived in this section, we can write the weight and charge distribution of our sequences. It seems to us that mostly convenient form for presenting this distribution is a triangle table, like Pascal's triangle. An example of such distribution is shown in Table III.

We may consider (6) as an implicit mutual recursion. Indeed, an element in left slanting row in the triangle of charge distribution (see Table III(a)) depends on elements in right slanting rows and vice versa. In more details, this will be shown in the next section.

III. DIRECT EQUATIONS FOR THE NUMBER OF SEQUENCES

Consider dkr - limited sequences. The problem of defining the number of constant-weight sequences, in which term's coefficients are Fibonacci numbers, i.e., for d - sequences, has been solved by Riordan [11]. He presented a direct, not recursion method for calculating the number of constant-weight d - sequences. Lee in [8] and Ytrehus in [6] extended Riordan's method for the dkr - constrained sequences. Our method of deriving the similar direct equations rests on recursions from the previous section.

A. Calculating the number of constant-weight sequences

From (1) we have that

$$A_1^1 = 1, A_2^1 = 1, \ldots, A_{r+1}^1 = 1, A_{r+2}^1 = 0, \ldots$$

or

$$A_n^1 = \binom{n-1}{0} - \binom{n-1-(r+1)}{0}$$

(suppose that $\binom{n}{k} = 0$ if $k > n$). From recursion relation (2) we have

$$\begin{aligned}
A_n^2 &= A_{n-(d+1)}^1 + A_{n-(d+2)}^1 + \cdots + A_{n-(k+1)}^1, \\
A_n^3 &= A_{n-(d+1)}^2 + A_{n-(d+2)}^2 + \cdots + A_{n-(k+1)}^2, \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
A_n^\nu &= A_{n-(d+1)}^{\nu-1} + A_{n-(d+2)}^{\nu-1} + \cdots + A_{n-(k+1)}^{\nu-1}.
\end{aligned}$$

Substituting A_n^1 into A_n^2, we obtain

$$\begin{aligned}
A_n^2 = {} & \tbinom{n-(d+1)-1}{0} - \tbinom{n-(d+1)-1-(r+1)}{0} \\
& + \tbinom{n-(d+2)-1}{0} - \tbinom{n-(d+2)-1-(r+1)}{0} \\
& \cdots + \tbinom{n-(k+1)-1}{0} - \tbinom{n-(k+1)-1-(r+1)}{0} \\
= {} & \tbinom{n-(d+1)}{1} - \tbinom{n-(r+1)-(d+1)}{1} \\
& - \left(\tbinom{n-1-(k+1)}{1} - \tbinom{n-1-(r+1)-(k+1)}{1} \right).
\end{aligned}$$

Likewise,

$$\begin{aligned}
A_n^3 &= \tbinom{n+1-2(d+1)}{2} - \tbinom{n+1-(r+1)-2(d+1)}{2} \\
&\quad - 2\left(\tbinom{n-(d+1)-(k+1)}{2} - \tbinom{n-(r+1)-(d+1)-(k+1)}{2}\right) \\
&\quad + \tbinom{n-1-2(k+1)}{2} - \tbinom{n-1-(r+1)-2(k+1)}{2}, \\
\vdots \quad & \quad \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
A_n^\nu &= \sum_{j=0}^{\nu-1} (-1)^j \tbinom{\nu-1}{j} \left(\tbinom{n-1+(\nu-1-j)-(\nu-1-j)(d+1)-j(k+1)}{\nu-1}\right. \\
&\quad \left. - \tbinom{n-1+(\nu-1-j)-(r+1)-(\nu-1-j)(d+1)-j(k+1)}{\nu-1}\right), \\
&\qquad \nu \geq 1. \qquad (8)
\end{aligned}$$

By q denote the number of possible lengths of runs beginning with one

$$q = k - d + 1.$$

Then we can rewrite (8) in more convenient form

$$\begin{aligned}
A_n^\nu = \sum_{j=0}^{\nu-1} (-1)^j \tbinom{\nu-1}{j} &\left(\tbinom{n-1-(\nu-1)d-jq}{\nu-1}\right. \\
&\left. - \tbinom{n-1-(r+1)-(\nu-1)d-jq}{\nu-1}\right), \quad \nu \geq 1. \qquad (9)
\end{aligned}$$

B. Calculating the number of constant-charge sequences

Now, consider the constant-charge sequences. For these sequences we may simply show that

$$C_0^0 = 1, C_1^{-1} = 1, \ldots, C_{r+1}^{-(r+1)} = 1, C_{r+2}^{-(r+2)} = 0, \ldots$$

or

$$C_n^{-n} = \binom{n}{0} - \binom{n-1-(r+1)}{0}. \qquad (10)$$

This directly follows from initial conditions (5) of recursion relation (6).

Thus, we define the first left slanting row in our triangle of numbers of constant-charge sequences. For example, see Table III(a).

We also may show that for right slanting rows

$$\begin{array}{llll}
C_0^0 = 1, & C_1^1 = 0, & C_2^2 = 0, \ldots & d \geq 0, \\
C_1^{-1} = 1, & C_2^0 = 0, & C_3^1 = 0, \ldots & d \geq 1, \\
\vdots & \vdots & \vdots & \vdots \\
C_{r+1}^{-r+1} = 1, & C_{r+2}^{-(r+1)+1} = 0, & C_{r+3}^{-(r+1)+2} = 0, \ldots & d \geq r+1, \\
C_{r+2}^{-r+2} = 0, & C_{r+3}^{-(r+2)+1} = 0, & C_{r+4}^{-(r+2)+2} = 0, \ldots &
\end{array} \qquad (11)$$

Let δ be an even number which shows how the charge σ differs from n or from $-n$. In other words, $\delta/2$ is an index of a slanting row in the triangle of numbers of constant-charge sequences.

Therefore, we can rewrite (10) as

$$\begin{aligned}
C_n^{-n+\delta} = &\left(\binom{\delta/2}{0} - \binom{\delta/2-1}{0}\right) \\
&\times \left(\binom{n}{0} - \binom{n-1-(r+1)}{0}\right). \qquad (12)
\end{aligned}$$

Also we can rewrite (11) as

$$\begin{aligned}
C_n^{n-\delta} = &\left(\binom{\delta/2-1}{0} - \binom{\delta/2-1-(r+1)}{0}\right) \\
&\times \left(\binom{n-\delta/2}{0} - \binom{n-\delta/2-1}{0}\right), \quad d \geq \delta/2. \qquad (13)
\end{aligned}$$

If $d < \delta/2$ then for right slanting rows, we have from recursion relation (6) that

$$\begin{aligned} C_n^{n-\delta} &= C_{n-(d+1)}^{-n+\delta-(d+1)} + C_{n-(d+2)}^{-n+\delta-(d+2)} + \cdots + C_{n-(k+1)}^{-n+\delta-(k+1)} \\ &= C_{n-(d+1)}^{-(n-(d+1))+\delta-2(d+1)} + C_{n-(d+2)}^{-(n-(d+2))+\delta-2(d+2)} \\ &\cdots + C_{n-(k+1)}^{-(n-(k+1))+\delta-2(k+1)}. \end{aligned} \tag{14}$$

This confirms a mutual nature of (6). The series (14) terminates early if $\sigma < -n$, i.e., whenever someone of $\delta - 2(d+1)$, $\delta - 2(d+2)$, …, $\delta - 2(k+1)$ becomes less then 0. Therefore, combining (13) with (14), we obtain

$$\begin{aligned} C_n^{n-\delta} &= C_{n-(d+1)}^{-(n-(d+1))+\delta-2(d+1)} + C_{n-(d+2)}^{-(n-(d+2))+\delta-2(d+2)} \\ &\cdots + C_{n-(k+1)}^{-(n-(k+1))+\delta-2(k+1)} \\ &+ \left(\binom{\delta/2-1}{0} - \binom{\delta/2-1-(r+1)}{0}\right)\left(\binom{n-\delta/2}{0} - \binom{n-\delta/2-1}{0}\right) \\ &- \left(\binom{\delta/2-(d+1)}{0} - \binom{\delta/2-1-(d+1)}{0}\right. \\ &+ \binom{\delta/2-(d+2)}{0} - \binom{\delta/2-1-(d+2)}{0} \\ &\left.\cdots + \binom{\delta/2-(k+1)}{0} - \binom{\delta/2-1-(k+1)}{0}\right) \\ &\times \left(\binom{n-\delta/2}{0} - \binom{n-\delta/2-1}{0}\right), \quad \delta > 0, \end{aligned}$$

then

$$\begin{aligned} C_n^{n-\delta} &= C_{n-(d+1)}^{-(n-(d+1))+\delta-2(d+1)} + C_{n-(d+2)}^{-(n-(d+2))+\delta-2(d+2)} \\ &\cdots + C_{n-(k+1)}^{-(n-(k+1))+\delta-2(k+1)} \\ &+ \left(\binom{\delta/2-1}{0} - \binom{\delta/2-(d+1)}{0}\right. \\ &\left. + \binom{\delta/2-1-(k+1)}{0} - \binom{\delta/2-1-(r+1)}{0}\right) \\ &\times \left(\binom{n-\delta/2}{0} - \binom{n-\delta/2-1}{0}\right), \quad \delta > 0. \end{aligned} \tag{15}$$

Consider an example. For $\delta = 2$ and for $\sigma = n - \delta$, we have from recursion relation (6) that

$$\begin{aligned} C_n^{n-2} &= C_{n-(d+1)}^{-n+2-(d+1)} + C_{n-(d+2)}^{-n+2-(d+2)} + \cdots + C_{n-(k+1)}^{-n+2-(k+1)} \\ &= C_{n-(d+1)}^{-(n-(d+1))+2-2(d+1)} + C_{n-(d+2)}^{-(n-(d+2))+2-2(d+2)} \\ &\cdots + C_{n-(k+1)}^{-(n-(k+1))+2-2(k+1)} \end{aligned}$$

or

$$C_n^{n-2} = C_{n-1}^{-(n-1)}, \quad d = 0.$$

Since this equation specifies the right slanting rows, then we have to substitute (10) for $C_{n-1}^{-(n-1)}$, if $d = 0$, and (13) otherwise. Therefore we obtain the next result

$$C_n^{n-2} = \begin{cases} \binom{n-1}{0} - \binom{n-1-1-(r+1)}{0}, & d = 0, \\ \binom{n-1}{0} - \binom{n-1-1}{0}, & \text{otherwise.} \end{cases}$$

Using (15), we can rewrite this piecewise-continuous equation as

$$\begin{aligned} C_n^{n-2} = &\left(\binom{1+1-(d+1)}{1} - \binom{1-(d+1)}{1}\right. \\ &\left. - \left(\binom{1-(k+1)}{1} - \binom{1-1-(k+1)}{1}\right)\right) \\ &\times \left(\binom{n-1}{0} - \binom{n-1-1-(r+1)}{0}\right) \\ &+ \left(\binom{1-1}{0} - \binom{1-(d+1)}{0} + \binom{1-1-(k+1)}{0} - \binom{1-1-(r+1)}{0}\right) \\ &\times \left(\binom{n-1}{0} - \binom{n-1-1}{0}\right). \end{aligned}$$

For $\delta = 2$ and for $\sigma = -n + \delta$, we have from the same recursion relation

$$C_n^{-n+2} = C_{n-(d+1)}^{n-(d+1)-2} + C_{n-(d+2)}^{n-(d+2)-2} + \cdots + C_{n-(k+1)}^{n-(k+1)-2}$$

and obtain

$$\begin{aligned}C_n^{-n+2} = &\Big(\tbinom{1+1-(d+1)}{1} - \tbinom{1-(d+1)}{1} \\ &- \Big(\tbinom{1-(k+1)}{1} - \tbinom{1-1-(k+1)}{1}\Big)\Big) \\ &\times \Big(\tbinom{n-1+1-(d+1)}{1} - \tbinom{n-1-(r+1)-(d+1)}{1} \\ &- \Big(\tbinom{n-1-(k+1)}{1} - \tbinom{n-1-1-(r+1)-(k+1)}{1}\Big)\Big) \\ &+ \Big(\tbinom{1-1}{0} - \tbinom{1-(d+1)}{0} + \tbinom{1-1-(k+1)}{0} - \tbinom{1-1-(r+1)}{0}\Big) \\ &\times \Big(\tbinom{n-1+1-(d+1)}{1} - \tbinom{n-1-(d+1)}{1} \\ &- \Big(\tbinom{n-1-(k+1)}{1} - \tbinom{n-1-1-(k+1)}{1}\Big)\Big).\end{aligned}$$

Let A be a value which has the similar combinatorial meaning as A in (2) and let

$$\begin{aligned}A_\delta^m = \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \Big(&\tbinom{\delta/2+m-j-(m-j)(d+1)-j(k+1)}{m} \\ &- \tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)}{m}\Big)\end{aligned} \tag{16}$$

and

$$\begin{aligned}\widetilde{A}_\delta^m = \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \Big(&\tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)}{m} \\ &- \tbinom{\delta/2+m-j-(d+1)-(m-j)(d+1)-j(k+1)}{m} \\ &+ \tbinom{\delta/2+m-j-1-(k+1)-(m-j)(d+1)-j(k+1)}{m} \\ &- \tbinom{\delta/2+m-j-1-(r+1)-(m-j)(d+1)-j(k+1)}{m}\Big).\end{aligned} \tag{17}$$

Also, let

$$\begin{aligned}B_{n,\delta}^m = \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \Big(&\tbinom{n-\delta/2+m-j-(m-j)(d+1)-j(k+1)}{m} \\ &- \tbinom{n-\delta/2+m-j-1-(r+1)-(m-j)(d+1)-j(k+1)}{m}\Big)\end{aligned} \tag{18}$$

and

$$\begin{aligned}\widetilde{B}_{n,\delta}^m = \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \Big(&\tbinom{n-\delta/2+m-j-(m-j)(d+1)-j(k+1)}{m} \\ &- \tbinom{n-\delta/2+m-j-1-(m-j)(d+1)-j(k+1)}{m}\Big).\end{aligned} \tag{19}$$

Using this notation, we may rewrite C_n^{-n} and C_n^{-n+2} from our examples as

$$\begin{aligned}C_n^{-n} &= A_0^0 B_{n,0}^0, \\ C_n^{-n+2} &= A_2^1 B_{n,2}^1 + \widetilde{A}_2^0 \widetilde{B}_{n,2}^1.\end{aligned}$$

Likewise,

$$C_n^{-n+4} = A_4^1 B_{n,4}^1 + A_4^2 B_{n,4}^2 + \widetilde{A}_4^0 \widetilde{B}_{n,4}^1 + \widetilde{A}_4^1 \widetilde{B}_{n,4}^2.$$

Continuing in the same way, we see that

$$C_n^{-n+\delta} = \sum_{m=0}^{\delta/2} \Big(A_\delta^m B_{n,\delta}^m + \widetilde{A}_\delta^{m-1} \widetilde{B}_{n,\delta}^m\Big). \tag{20}$$

Observe that the upper limit of summation might be written as a quotient of δ and $2(d+1)$. In such case

$$C_n^{-n+\delta} = \sum_{m=0}^{\left\lceil \frac{\delta}{2(d+1)} \right\rceil} \left(A_\delta^m B_{n,\delta}^m + \widetilde{A}_\delta^{m-1} \widetilde{B}_{n,\delta}^m \right). \tag{21}$$

Note that summing $C_n^{-n+\delta}$ in right slanting rows does not affect $B_{n,\delta}^m$ and $\widetilde{B}_{n,\delta}^m$. This affects only A_δ^m and $\widetilde{A}_\delta^m$. Indeed, from (15) it follows that index of summation, which subtracts from n and from $\delta/2$, annihilates in (18) and (19) and remains in (16) and (17).

Now, consider $\sum_{i=d+1}^{k+1} A_{\delta-i}^m$. We have

$$\begin{aligned}
\sum_{i=d+1}^{k+1} A_{\delta-i}^m = \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} & \\
\times \Big(& \tbinom{\delta/2+m-j-(m-j)(d+1)-j(k+1)-(d+1)}{m} \\
& - \tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)-(d+1)}{m} \\
& + \tbinom{\delta/2+m-j-(m-j)(d+1)-j(k+1)-(d+2)}{m} \\
& - \tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)-(d+2)}{m} \\
& \cdots + \tbinom{\delta/2+m-j-(m-j)(d+1)-j(k+1)-(k+1)}{m} \\
& - \tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)-(k+1)}{m} \Big) \\
= \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \Big(& \tbinom{\delta/2+m-j-(m-j)(d+1)-j(k+1)+1-(d+1)}{m+1} \\
& - \tbinom{\delta/2+m-j-(m-j)(d+1)-j(k+1)-(k+1)}{m+1} \\
& - \Big(\tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)+1-(d+1)}{m+1} \\
& - \tbinom{\delta/2+m-j-1-(m-j)(d+1)-j(k+1)-(k+1)}{m+1} \Big) \Big) \\
= \sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \Big(& \tbinom{\delta/2+m+1-j-(m+1-j)(d+1)-j(k+1)}{m+1} \\
& - \tbinom{\delta/2+m+1-j-1-(m+1-j)(d+1)-j(k+1)}{m+1} \Big) \\
+ \sum_{j=1}^{m+1} (-1)^j \tbinom{m}{j-1} \Big(& \tbinom{\delta/2+m+1-j-(m+1-j)(d+1)-j(k+1)}{m+1} \\
& - \tbinom{\delta/2+m+1-j-1-(m+1-j)(d+1)-j(k+1)}{m+1} \Big) \\
= \sum_{j=0}^{m+1} (-1)^j \tbinom{m+1}{j} \Big(& \tbinom{\delta/2+m+1-j-(m+1-j)(d+1)-j(k+1)}{m+1} \\
& - \tbinom{\delta/2+m+1-j-1-(m+1-j)(d+1)-j(k+1)}{m+1} \Big).
\end{aligned} \tag{22}$$

This means that

$$\sum_{i=d+1}^{k+1} A_{\delta-i}^m = A_\delta^{m+1}.$$

Similarly,

$$\sum_{i=d+1}^{k+1} \widetilde{A}_{\delta-i}^m = \widetilde{A}_\delta^{m+1}.$$

Using this result and using (15) again we see that

$$C_n^{n-\delta} = \sum_{m=0}^{\delta/2} \left(A_\delta^{m+1} B_{n,\delta}^m + \widetilde{A}_\delta^m \widetilde{B}_{n,\delta}^m \right), \quad \delta > 0$$

and like (21), we may rewrite

$$C_n^{n-\delta}=\sum_{m=0}^{\left\lceil\frac{\delta}{2(d+1)}\right\rceil}\left(A_\delta^{m+1}B_{n,\delta}^m+\widetilde{A}_\delta^m\widetilde{B}_{n,\delta}^m\right),\quad \delta>0.$$

For $\sigma=-n+\delta$, we have from recursion relation (6) that

$$C_n^{-n+\delta}=C_{n-(d+1)}^{n-(d+1)-\delta}+C_{n-(d+2)}^{n-(d+2)-\delta}+\cdots+C_{n-(k+1)}^{n-(k+1)-\delta}. \tag{23}$$

Similarly, we can see that summing $C_n^{n-\delta}$ in left slanting rows does not affect A_δ^m and $\widetilde{A}_\delta^m$. This affects $B_{n,\delta}^m$ and $\widetilde{B}_{n,\delta}^m$. Namely, from (23) it follows that index of summation subtracts only from n.

Consider $\sum_{i=d+1}^{k+1}B_{n-i,\delta}^m$. Applying a reasoning chain like (22) yields

$$\begin{aligned}\sum_{i=d+1}^{k+1}B_{n-i,\delta}^m&=\sum_{j=0}^{m+1}(-1)^j\tbinom{m+1}{j}\\&\times\left(\tbinom{n-\delta/2+m+1-j-(m+1-j)(d+1)-j(k+1)}{m+1}\right.\\&-\left.\tbinom{n-\delta/2+m+1-j-1-(r+1)-(m+1-j)(d+1)-j(k+1)}{m+1}\right).\end{aligned}$$

This means that

$$\sum_{i=d+1}^{k+1}B_{n-i,\delta}^m=B_{n,\delta}^{m+1}.$$

Similarly,

$$\sum_{i=d+1}^{k+1}\widetilde{B}_{n-i,\delta}^m=\widetilde{B}_{n,\delta}^{m+1}.$$

Using this result and using (23) we obtain

$$C_n^{-n+\delta}=\sum_{m=0}^{\delta/2}\left(A_\delta^{m+1}B_{n,\delta}^{m+1}+\widetilde{A}_\delta^m\widetilde{B}_{n,\delta}^{m+1}\right),\quad \delta>0. \tag{24}$$

This proves the inductive hypothesis (20).

Like (21), we may rewrite (24) as

$$C_n^{-n+\delta}=\sum_{m=0}^{\left\lceil\frac{\delta}{2(d+1)}\right\rceil}\left(A_\delta^{m+1}B_{n,\delta}^{m+1}+\widetilde{A}_\delta^m\widetilde{B}_{n,\delta}^{m+1}\right),\quad \delta>0. \tag{25}$$

Substituting $n+\sigma$ for δ in (21), we get (26) at the bottom of the page.

$$\begin{aligned}C_n^\sigma=\sum_{m=0}^{\left\lceil\frac{n+\sigma}{2(d+1)}\right\rceil}&\left(\left(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}\left(\tbinom{\frac{n+\sigma}{2}-md-jq}{m}-\tbinom{\frac{n+\sigma}{2}-1-md-jq}{m}\right)\right)\right.\\&\times\left(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}\left(\tbinom{\frac{n-\sigma}{2}-md-jq}{m}-\tbinom{\frac{n-\sigma}{2}-1-(r+1)-md-jq}{m}\right)\right)\\&+\left(\sum_{j=0}^{m-1}(-1)^j\tbinom{m-1}{j}\left(\tbinom{\frac{n+\sigma}{2}-1-(m-1)d-jq}{m-1}-\tbinom{\frac{n+\sigma}{2}-(d+1)-(m-1)d-jq}{m-1}\right.\right.\\&+\left.\left.\tbinom{\frac{n+\sigma}{2}-1-(k+1)-(m+1)d-jq}{m-1}-\tbinom{\frac{n+\sigma}{2}-1-(r+1)-(m-1)d-jq}{m-1}\right)\right)\\&\times\left.\left(\sum_{j=0}^{m}(-1)^j\tbinom{m}{j}\left(\tbinom{\frac{n-\sigma}{2}-md-jq}{m}-\tbinom{\frac{n-\sigma}{2}-1-md-jq}{m}\right)\right)\right)\end{aligned} \tag{26}$$

IV. Generating Functions for Enumerating the Constant-Weight and Constant-Charge Sequences

From triangle table (see Table III(a)) it follows that there exist two types of sequences A_n^ν and C_n^σ. The infinite sequences of the first type are $A_0^\nu, A_1^\nu, \ldots$ then $C_0^\sigma, C_2^\sigma, \ldots$ or $C_1^\sigma, C_3^\sigma, \ldots$. The finite sequences of the second type are $A_n^1, A_n^2, \ldots, A_n^n$ and $C_n^{-n}, C_n^{-n+2}, \ldots, C_n^n$. Kolesnik and Krachkovsky described in [12] a recursive calculation of generating function for enumerating constant-weight sequences of the first type. Lee in [8] suggested a direct method for the same. Here we obtain the generating functions of the both types.

A. Generating Function of Sequence $A_0^\nu, A_1^\nu, \ldots$

Consider infinite sequence $A_0^\nu, A_1^\nu, A_2^\nu, \ldots$. Define generating function of this sequence as

$$A^\nu(t) = \sum_{n=0}^{\infty} A_n^\nu t^n, \quad t \in \mathbb{R}.$$

Substituting (9) for A_n^ν here and changing the order of summation we have that

$$\begin{aligned} A^\nu(t) = &\sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \\ &\times \left(\sum_{\substack{n=(\nu-1)+1+(\nu-1)d\\+jq}}^{\infty} \binom{n-1-(\nu-1)d-jq}{\nu-1} t^n \right. \\ &\left. - \sum_{\substack{n=(\nu-1)+1+(\nu-1)d\\+(r+1)+jq}}^{\infty} \binom{n-1-(\nu-1)d-(r+1)-jq}{\nu-1} t^n \right). \end{aligned}$$

By using the fact that $\sum\limits_{n=\nu+a}^{\infty} \binom{n-a}{\nu} t^n = \left(t^\nu/(1-t)^{\nu+1}\right) t^a$ we get

$$A^\nu(t) = \frac{t^{\nu-1}}{(1-t)^\nu} t^{1+(\nu-1)d}(1-t^{r+1})(1-t^q)^{\nu-1}. \tag{27}$$

B. Generating Function of Sequence $A_n^1, A_n^2, \ldots, A_n^n$

Consider a finite power series

$$A_n(y) = \sum_{\nu=1}^{n} A_n^\nu y^\nu, \quad y \in \mathbb{R}.$$

We can obtain $A_n(y)$ in closed form by the following way. Consider a formal power series

$$A(t, y) = \sum_{\nu=1}^{\infty} A^\nu(t) y^\nu.$$

We can achieve convergence of this series by assuming t arbitrarily small. Rewrite (27) as

$$A^\nu(t) = \left(\frac{t^{d+1}(1-t^q)}{1-t}\right)^{\nu-1} \frac{t(1-t^{r+1})}{1-t}.$$

Then we have

$$\begin{aligned} A(t, y) &= \frac{yt(1-t^{r+1})}{1-t} \sum_{\nu=1}^{\infty} \left(\frac{yt^{d+1}(1-t^q)}{1-t}\right)^{\nu-1} \\ &= \frac{yt(1-t^{r+1})}{1-t-yt^{d+1}+yt^{k+2}}. \end{aligned} \tag{28}$$

By using the fact that $dA(t, y)/dt = \sum\limits_{n=1}^{\infty} nA_n(y)t^{n-1}$ we obtain

$$A_n(y) = \frac{1}{n!} \lim_{t \to 0} \frac{d^n A(t, y)}{dt^n}.$$

From Cauchy's integral representation, we write

$$\frac{d^n A(t,y)}{dt^n} = \frac{n!}{2\pi i}\oint_{\Gamma} \frac{y\tau(1-\tau^{r+1})}{(1-\tau-y\tau^{d+1}+y\tau^{k+2})\,(\tau-t)^{n+1}}d\tau.$$

Then

$$A_n(y) = \frac{y}{2\pi i}\oint_{\Gamma} \frac{1-\tau^{r+1}}{(1-\tau-y\tau^{d+1}+y\tau^{k+2})\,\tau^{n}}d\tau, \tag{29}$$

where Γ is a counterclockwise simple closed contour surrounding the origin of the complex plane, small enough to avoid any other poles of integrand in (29). The application of the residue theorem yields

$$A_n(y) = \sum_{j=1}^{k+2} \frac{\tau_j^{r+1}-1}{\tau_j^n \prod\limits_{\substack{m=1,\\ m\neq j}}^{k+2} (\tau_j-\tau_m)}, \tag{30}$$

where $\tau_1,\, \tau_2,\, \ldots,\, \tau_{k+2}$ are roots of $1-\tau-y\tau^{d+1}+y\tau^{k+2}$ and $\tau_1 = 1$.

C. *Generating Function of Sequences $C_0^\sigma, C_2^\sigma, \ldots$ or $C_1^\sigma, C_3^\sigma, \ldots$*

Similarly, consider an infinite sequence $C_0^\sigma, C_1^\sigma, C_2^\sigma, \ldots$. In such case, we define generating function as

$$C^\sigma(t) = \sum_{\substack{n=0\\ n \text{ even}}}^{\infty} C_n^\sigma t^{n/2}, \quad \text{or} \quad C^\sigma(t) = \sum_{\substack{n=1\\ n \text{ odd}}}^{\infty} C_n^\sigma t^{(n-1)/2}, \tag{31}$$

where $t \in \mathbb{R}$.

It is easily shown that (26) can be written as a sum of 12 terms with proper signs.

$$C_n^\sigma = \sum_{\iota=1}^{12} (-1)^{\lfloor \iota/2 \rfloor} C(\iota)_n^\sigma. \tag{32}$$

Define these terms $C(\iota)_n^\sigma$ as

$$\begin{aligned} C(\iota)_n^\sigma = \sum_{m=0}^{\left\lceil \frac{n+\sigma}{2(d+1)} \right\rceil} &\left(\sum_{j=0}^{m-\mu} (-1)^j \tbinom{m-\mu}{j} \right. \\ &\qquad \left. \times \tbinom{\frac{n+\sigma}{2}-p_1-md-jq-\mu}{m-\mu} \right) \\ &\times \left(\sum_{j=0}^{m} (-1)^j \tbinom{m}{j} \tbinom{\frac{n-\sigma}{2}-p_2-md-jq}{m} \right), \end{aligned} \tag{33}$$

where

$$p_1 = \begin{cases} 0, & I = 1,2,7,8, \\ 1, & I = 3,4, \\ -d, & I = 5,6, \\ (k+1)-d, & I = 9,10, \\ (r+1)-d, & I = 11,12, \end{cases}$$

$$p_2 = \begin{cases} 0, & I = 1,3,5,7,9,11, \\ 1+(r+1), & I = 2,4, \\ 1, & I = 6,8,10,12, \end{cases}$$

and

$$\mu = \begin{cases} 0, & I = 1,2,3,4, \\ 1, & I = 5,6,7,8,9,10,11,12. \end{cases}$$

Without loss of generality we will consider a term of (32) as C_n^σ and assume that $C_n^\sigma := C(\iota)_n^\sigma$ below in this section. Of course, this implies using (32) when calculating.

We can expand the inner product of two series using double sums as follows:

$$\left(\sum_{j=0}^{m-\mu}(-1)^j\binom{m-\mu}{j}\binom{\frac{n+\sigma}{2}-p_1-md-jq-\mu}{m-\mu}\right)$$
$$\times\left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}\binom{\frac{n-\sigma}{2}-p_2-md-jq}{m}\right)$$
$$=\sum_{u=0}^{m-\mu}\sum_{v=0}^{m}(-1)^{u+v}\binom{m-\mu}{u}\binom{m}{v}$$
$$\times\binom{\frac{n+\sigma}{2}-p_1-md-uq-\mu}{m-\mu}\binom{\frac{n-\sigma}{2}-p_2-md-vq}{m};$$

then interchanging the sums, we get

$$C^\sigma(t)=\sum_{m=0}^{\infty}\left(\sum_{u=0}^{m-\mu}\sum_{v=0}^{m}(-1)^{u+v}\binom{m-\mu}{u}\binom{m}{v}g(t)\right), \tag{34}$$

where by $g(t)$ we denote the inner sum.

$$g(t)=\sum_{\substack{n=\rho\\ n\ \text{even}\\ \text{or}\\ n\ \text{odd}}}^{\infty}\binom{\frac{n+\sigma}{2}-p_1-md-uq-\mu}{m-\mu}\binom{\frac{n-\sigma}{2}-p_2-md-vq}{m}t^{(n-\rho)/2}, \tag{35}$$

where $\rho=\sigma \bmod 2$.

Further, we will derive the generating function using arguing style of orthogonal polynomials theory [13].

Let $G(n)=\binom{\frac{n+\sigma}{2}-p_1-md-uq-\mu}{m-\mu}\binom{\frac{n-\sigma}{2}-p_2-md-vq}{m}t^{(n-\rho)/2}$, $m-\mu\geq 0$. The sequence $G(n)$ may content leading zero elements. In this case we can not write a ratio between consecutive terms. To prevent this we can use the following rule

$$\sum_{j=0}^{\infty}\binom{j-A}{B}\binom{j-C}{D}=\begin{cases}\sum\limits_{j=A+B}^{\infty}\binom{j-A}{B}\binom{j-C}{D}, & A+B-C\geq D,\\ \sum\limits_{j=C+D}^{\infty}\binom{j-A}{B}\binom{j-C}{D}, & C+D-A\geq B.\end{cases}$$

Let α be the $A+B-C-D$ in the conditionals above. In our case $\alpha=-\sigma+p_1-p_2+(u-v)q$ and we can rewrite (35) as

$$g(t)=t^{m(d+1)+a}$$
$$\times\sum_{j=0}^{\infty}\binom{m-\beta+j}{m-\beta}\binom{m-\mu+\beta+|\alpha|+j}{m-\mu+\beta}t^j, \tag{36}$$

where

$$a=\begin{cases}\lfloor-\sigma/2\rfloor+p_1+uq, & \alpha\geq 0,\\ \lfloor\sigma/2\rfloor+p_2+vq, & \text{otherwise}\end{cases}$$

and

$$\beta=\begin{cases}\mu, & \alpha\geq 0,\\ 0, & \text{otherwise.}\end{cases}$$

Now, consider the sum in (36). By $G(j)$ denote a term of this series $\binom{m-\beta+j}{m-\beta}\binom{m-\mu+\beta+|\alpha|+j}{m-\mu+\beta}t^j$ and write the term ratio as

$$\frac{G(j+1)}{G(j)}=\frac{\binom{m-\beta+j+1}{m-\beta}\binom{m-\mu+\beta+|\alpha|+j+1}{m-\mu+\beta}t^{j+1}}{\binom{m-\beta+j}{m-\beta}\binom{m-\mu+\beta+|\alpha|+j}{m-\mu+\beta}t^j}$$
$$=\frac{(j+2)_{m-\beta}(|\alpha|+j+2)_{m-\mu+\beta}t}{(j+1)_{m-\beta}(|\alpha|+j+1)_{m-\mu+\beta}}$$
$$=\frac{(m-\beta+j+1)(m-\mu+\beta+|\alpha|+j+1)t}{(j+1)(|\alpha|+j+1)},$$

where $(a)_j$ denote the Pochhammer-symbol which is defined by $(a)_0 = 1$ and $(a)_j = a(a+1)(a+2)\cdots(a+j-1)$, $j = 1, 2, 3, \ldots$ Obviously that $G(0) = \binom{m-\mu+\beta+|\alpha|}{m-\mu+\beta}$ and hence,

$$G(j) = G(0)\frac{(m-\beta+1)_j(m-\mu+\beta+|\alpha|+1)_j}{(|\alpha|+1)_j}\frac{t^j}{j!}.$$

By summing $G(j)$ we get the hypergeometric series as follows:

$$\begin{aligned} g(t) &= t^{m(d+1)+a}G(0) \\ &\qquad \times {}_2F_1\left(\begin{matrix} m-\beta+1, m-\mu+\beta+|\alpha|+1 \\ |\alpha|+1 \end{matrix}; t\right). \end{aligned}$$

If we apply the Pfaff-Euler transformation formula [14], then we obtain the next terminating series

$$g(t) = \frac{t^{m(d+1)+a}}{(1-t)^{2m-\mu+1}}G(0)\,{}_2F_1\left(\begin{matrix} -m+\mu-\beta, |\alpha|+\beta-m \\ |\alpha|+1 \end{matrix}; t\right).$$

By several transformations we get

$$\begin{aligned} g(t) &= \frac{t^{m(d+1)+a}}{(1-t)^{2m-\mu+1}}G(0)\sum_{j=0}^{m-\mu+\beta}\binom{m-\mu+\beta}{j} \\ &\times \frac{(m-\beta-|\alpha|)(m-\beta-|\alpha|-1)\cdots(m-\beta-|\alpha|-j+1)}{(|\alpha|+1)(|\alpha|+2)\cdots(|\alpha|+1+j-1)}t^j \end{aligned}$$

then, using the distributive law, we obtain

$$\begin{aligned} g(t) &= \frac{t^{m(d+1)+a}}{(1-t)^{2m-\mu+1}}\frac{1}{(m-\mu+\beta)!}\sum_{j=0}^{m-\mu+\beta}\binom{m-\mu+\beta}{j} \\ &\times (m-\mu+\beta+|\alpha|)(m-\mu+\beta+|\alpha|-1)\cdots(|\alpha|+j+1) \\ &\times (m-\beta-|\alpha|)(m-\beta-|\alpha|-1)\cdots(m-\beta-|\alpha|-j+1)t^j. \end{aligned}$$

Now, our goal is to apply the Leibniz rule to this finite series. We have a power series in one variable t. But, we need a power series in two variables because we have rising and falling factorial powers here. Recall the Möbius transformation and suppose that

$$t = \frac{x-1}{x+1}, \quad \text{(assume } x \in \mathbb{R} \text{ till the end of this section)} \tag{37}$$

and rewrite

$$\begin{aligned} g(t) &= t^{m(d+1)+a-|\alpha|}\frac{(x+1)^{m-\mu+\beta+1}}{2^{2m-\mu+1}} \\ &\quad \times \frac{1}{(m-\mu+\beta)!}\sum_{j=0}^{m-\mu+\beta}\binom{m-\mu+\beta}{j} \\ &\quad \times (m-\mu+\beta+|\alpha|)(m-\mu+\beta+|\alpha|-1)\cdots \\ &\quad \times (m-\mu+\beta+|\alpha|-(m-\mu+\beta-j)+1) \\ &\quad \times (x-1)^{m-\mu+\beta+|\alpha|-(m-\mu+\beta-j)} \\ &\quad \times (m-\beta-|\alpha|)(m-\beta-|\alpha|-1)\cdots \\ &\quad \times (m-\beta-|\alpha|-j+1)(x+1)^{m-\beta-|\alpha|-j}. \end{aligned}$$

So, we obtain $(m-\mu+\beta-j)$th derivative of $(x+1)^{m-\mu+\beta+1}$ and (j)th derivative of $(x+1)^{m-\beta-|\alpha|}$.

$$\begin{aligned} g(t) &= t^{m(d+1)+\bar{a}}\frac{(x+1)^{m-\mu+\beta+1}}{2^{2m-\mu+1}}\frac{1}{(m-\mu+\beta)!} \\ &\times \sum_{j=0}^{m-\mu+\beta}\binom{m-\mu+\beta}{j}\frac{d^{m-\mu+\beta-j}(x-1)^{m-\mu+\beta+|\alpha|}}{dx^{m-\mu+\beta-j}} \\ &\qquad \times \frac{d^j(x+1)^{m-\beta-|\alpha|}}{dx^j}, \end{aligned}$$

TABLE IV
NOMENCLATURE FOR (47)

The first double sum: $\sum_{u=0}^{\infty}\sum_{w=0}^{\infty}$	The second double sum: $\sum_{v=1}^{\infty}\sum_{w=0}^{\infty}$
$\alpha_u = -\sigma + p_1 - p_2 + uq,$ (41)	$\alpha_v = -\sigma + p_1 - p_2 - vq,$ (42)
$a_u = \lfloor \sigma/2 \rfloor + p_2 + \begin{cases} 0, & \alpha_u \geq 0, \\ \alpha_u, & \text{else}, \end{cases}$ (43)	$a_v = \lfloor -\sigma/2 \rfloor + p_1 - \begin{cases} \alpha_v, & \alpha_v \geq 0, \\ 0, & \text{else}, \end{cases}$ (44)
such that $\bar{a} = wq + a_u$. $\beta_u = \begin{cases} \mu, & \alpha_u \geq 0, \\ 0, & \text{else}, \end{cases}$ (45)	such that $\bar{a} = wq + a_v$. $\beta_v = \begin{cases} \mu, & \alpha_v \geq 0 \\ 0, & \text{else}. \end{cases}$ (46)

where $\bar{a} = a - |\alpha|$. Finally, we have the Rodrigues' formula

$$g(t) = t^{\bar{a}} \frac{(x+1)^{\beta+1-\mu}}{2^{1-\mu}} b_d^m \frac{1}{(m-\mu+\beta)!} \times \frac{d^{m-\mu+\beta}\left(\left(\frac{x-1}{x+1}\right)^{|\alpha|+\beta} \frac{(x^2-1)^m}{(x-1)^\mu}\right)}{dx^{m-\mu+\beta}},$$

where

$$b_d = \left(\frac{x-1}{x+1}\right)^{d+1} \frac{x+1}{4} = \frac{t^{d+1}}{2(1-t)}. \tag{38}$$

Since we suppose (37), then by $\tau = \frac{\xi-1}{\xi+1}$ we denote the similar mapping in the complex plane. From Cauchy's integral representation, we write

$$g(t) = t^{\bar{a}} \frac{(x+1)^{\beta+1-\mu}}{2^{1-\mu}} b_d^m \times \frac{1}{2\pi i} \oint_\gamma \frac{\tau^{|\alpha|+\beta}(\xi^2-1)^m}{(\xi-1)^\mu(\xi-x)^{m-\mu+\beta+1}} d\xi, \tag{39}$$

where γ is some Jordan curve about the point $x = \frac{1+t}{1-t}$; this point lies on the real axis.

Here, we conclude the principal part of our derivation. This sequence of transformation also is need for the sequel, at least twice.

Now, recall (34). Interchanging the order of summation and integration; then interchanging the order of summation (with proper justification) yields

$$C^\sigma(t) = \left(\frac{x+1}{2}\right)^{1-\mu} \frac{1}{2\pi i} \oint_\gamma \left(\sum_{u=0}^{\infty} \sum_{v=0}^{\infty} (-1)^{u+v} \times t^{\bar{a}}(x+1)^\beta \frac{\tau^{|\alpha|+\beta}}{(\xi-1)^\mu(\xi-x)^{\beta-\mu+1}} \times \sum_{m=\max(u+\mu,v)}^{\infty} \binom{m-\mu}{u}\binom{m}{v} s^m \right) d\xi,$$

where

$$s = b_d \frac{\xi^2-1}{\xi-x}. \tag{40}$$

The next step is interchanging the outer sums order for diagonal summing as $\sum_{u=0}^{m-1}\sum_{v=0}^{m} X(u,v) = \sum_{u=0}^{m-1}\sum_{w=0}^{m-1-u} X(u+w,w) + \sum_{v=1}^{m}\sum_{w=0}^{m-v} X(w,v+w)$. Now, we have to redefine $u := u+w$ and $v := w$ for the first double sum; also $u := w$ and $v := v+w$ for the second double sum. Then, by α_u, α_v, a_u, a_v, β_u, and β_v we denote α, $\bar{a} - wq$, and β for the first and for the second double sums as shown in Table IV. Similarly, denote by $f_{u,w}(s)$ and by $f_{v,w}(s)$ the inner sums over m and

rewrite the result as

$$C^{\sigma}(t) = \left(\frac{x+1}{2}\right)^{1-\mu}$$
$$\times \frac{1}{2\pi i}\oint_{\gamma}\left(\sum_{u=0}^{\infty}(-1)^u\sum_{w=0}^{\infty}t^{a_u+wq}\right.$$
$$\times (x+1)^{\beta_u}\frac{\tau^{|\alpha_u|+\beta_u}}{(\xi-1)^{\mu}(\xi-x)^{\beta_u-\mu+1}}f_{u,w}(s)$$
$$+\sum_{v=1}^{\infty}(-1)^v\sum_{w=0}^{\infty}t^{a_v+wq}$$
$$\left.\times (x+1)^{\beta_v}\frac{\tau^{|\alpha_v|+\beta_v}}{(\xi-1)^{\mu}(\xi-x)^{\beta_v-\mu+1}}f_{v,w}(s)\right)d\xi, \tag{47}$$

where

$$f_{u,w}(s) = \sum_{m=u+w+\mu}^{\infty}\binom{m-\mu}{u+w}\binom{m}{w}s^m, \quad u \geq 0, \tag{48}$$

and

$$f_{v,w}(s) = \sum_{m=v+w}^{\infty}\binom{m-\mu}{w}\binom{m}{v+w}s^m, \quad v \geq 1. \tag{49}$$

It is easily shown that (48) and (49) are, in principal, similar to (36). Arguing as above (see (36) $\ldots$ (39)) we obtain

$$f_{u,w}(z) = \left(\frac{z-1}{z+1}\right)^{w}\frac{(z+1)^{w+1}}{2^{u+2w+1}}$$
$$\times \frac{1}{2\pi i}\oint_{\Gamma}\frac{(\zeta-1)^u\left(\frac{\zeta-1}{\zeta+1}\right)^{\mu}(\zeta^2-1)^w}{(\zeta-z)^{w+1}}d\zeta \tag{50}$$

and

$$f_{v,w}(z) = \left(\frac{z-1}{z+1}\right)^{w+\mu}\frac{(z+1)^{w+1}}{2^{v+2w+1}}$$
$$\times \frac{1}{2\pi i}\oint_{\Gamma}\frac{(\zeta-1)^v\left(\frac{\zeta+1}{\zeta-1}\right)^{\mu}(\zeta^2-1)^w}{(\zeta-z)^{w+1}}d\zeta, \tag{51}$$

where Γ is a closed contour surrounding the point $\zeta = z$. Here we suppose that

$$s = \frac{z-1}{z+1}, \quad \text{(assume } z \in \mathbb{C} \text{ till the end of this section).}$$

Substituting (50), and (51) for $f_{u,w}(s)$ and $f_{v,w}(s)$ in (47); then interchanging the order of summation over w and integration

with respect to ζ, we get

$$C^{\sigma}(t) = \left(\frac{x+1}{2}\right)^{1-\mu} \frac{1}{2\pi i} \oint_{\gamma} \left(\sum_{u=0}^{\infty} (-1)^u t^{a_u} \right.$$
$$\times (x+1)^{\beta_u} \frac{\tau^{|\alpha_u|+\beta_u}}{(\xi-1)^{\mu}(\xi-x)^{\beta_u-\mu+1}} \frac{z+1}{2^{u+1}}$$
$$\times \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\zeta-1)^u \left(\frac{\zeta-1}{\zeta+1}\right)^{\mu}}{\zeta - z} \sum_{w=0}^{\infty} \left(h \frac{\zeta^2-1}{\zeta-z}\right)^w d\zeta$$
$$+ \sum_{v=1}^{\infty} (-1)^v t^{a_v} (x+1)^{\beta_v}$$
$$\times \frac{\tau^{|\alpha_v|+\beta_v}}{(\xi-1)^{\mu}(\xi-x)^{\beta_v-\mu+1}} \left(\frac{z-1}{z+1}\right)^{\mu} \frac{z+1}{2^{v+1}}$$
$$\left. \times \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\zeta-1)^v \left(\frac{\zeta+1}{\zeta-1}\right)^{\mu}}{\zeta - z} \sum_{w=0}^{\infty} \left(h \frac{\zeta^2-1}{\zeta-z}\right)^w d\zeta \right) d\xi, \tag{52}$$

where

$$h = t^q \frac{z-1}{4}. \tag{53}$$

Denote by I_u and I_v the inner integrals as follows:

$$I_u = \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\zeta-1)^u \left(\frac{\zeta-1}{\zeta+1}\right)^{\mu}}{\zeta - z} \sum_{w=0}^{\infty} \left(h \frac{\zeta^2-1}{\zeta-z}\right)^w d\zeta \tag{54}$$

and

$$I_v = \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\zeta-1)^v \left(\frac{\zeta+1}{\zeta-1}\right)^{\mu}}{\zeta - z} \sum_{w=0}^{\infty} \left(h \frac{\zeta^2-1}{\zeta-z}\right)^w d\zeta. \tag{55}$$

Note that for

$$\left| h \frac{\zeta^2-1}{\zeta-z} \right| < 1,$$

we have convergent series in these integrals.

Consider (54). We can write

$$I_u = \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\zeta-1)^u \left(\frac{\zeta-1}{\zeta+1}\right)^{\mu}}{\zeta - z} \frac{1}{1 - h\dfrac{\zeta^2-1}{\zeta-z}} d\zeta$$
$$= \frac{1}{2\pi i} \oint_{\Gamma} \frac{(\zeta-1)^u \left(\frac{\zeta-1}{\zeta+1}\right)^{\mu}}{\zeta - z - h(\zeta^2-1)} d\zeta.$$

Consider the integrand

$$\eta_u(\zeta) = \frac{\chi_u(\zeta)}{\psi_u(\zeta)}$$
$$= \frac{(\zeta-1)^u \left(\frac{\zeta-1}{\zeta+1}\right)^{\mu}}{\zeta - z - h(\zeta^2-1)}.$$

We have denominator of the integrand $\psi_u(\zeta) = \big(\zeta - z - h(\zeta^2-1)\big)(\zeta+1)$ which has three zeros:

$$\zeta_{1,2} = \frac{1 \mp \sqrt{1-4hz+4h^2}}{2h} \tag{56}$$

and $\zeta_3 = -1$, where $\lim_{h\to 0} \zeta_1 = z$ and ζ_2 becomes infinite as $h \to 0$.

Hence, for sufficiently small h, one may suppose that there is just one zero: ζ_1 lying inside Γ, and the integrand has a simple pole inside the contour of integration with residue

$$\operatorname*{Res}_{\zeta=\zeta_1}(\eta_u(\zeta)) = \left.\frac{\chi_u(\zeta)}{\psi_u'(\zeta)}\right|_{\zeta=\zeta_1},$$

where $\psi_u'(\zeta) = 1-2h\zeta$.

Since $I_u = \mathrm{Res}_{\zeta=\zeta_1}(\eta_u(\zeta))$, then

$$I_u = \frac{\left(\dfrac{1-R}{2h}-1\right)^u \left(\dfrac{1-2h-R}{1+2h-R}\right)^\mu}{R}, \tag{57}$$

where

$$R = \sqrt{1-4hz+4h^2}. \tag{58}$$

Now, consider (55). We can rewrite

$$I_v = \frac{1}{2\pi i}\oint_\Gamma \frac{(\zeta-1)^v \left(\dfrac{\zeta+1}{\zeta-1}\right)^\mu}{\zeta - z - h(\zeta^2-1)} d\zeta.$$

Consider the integrand

$$\begin{aligned}\eta_v(\zeta) &= \frac{\chi_v(\zeta)}{\psi_v(\zeta)}\\ &= \frac{(\zeta-1)^v \left(\dfrac{\zeta+1}{\zeta-1}\right)^\mu}{\zeta - z - h(\zeta^2-1)}.\end{aligned}$$

Recall that $v \geq 1$, then we have denominator of the integrand $\psi_v(\zeta) = \zeta - z - h(\zeta^2-1)$. In this case we also have (56) as zeros of the denominator with the same properties. Then $I_v = \mathrm{Res}_{\zeta=\zeta_1}(\eta_v(\zeta))$ or

$$I_v = \frac{\left(\dfrac{1-R}{2h}-1\right)^v \left(\dfrac{1+2h-R}{1-2h-R}\right)^\mu}{R}. \tag{59}$$

By substituting (57) and (59) to (52) we have

$$\begin{aligned}C^\sigma(t) = &\left(\frac{x+1}{2}\right)^{1-\mu}\\ &\times \frac{1}{2\pi i}\oint_\gamma \Bigg(\frac{1}{\xi-1}\left(\frac{\xi-x}{\xi-1}\right)^\mu \frac{z+1}{2R}\left(\frac{1-2h-R}{1+2h-R}\right)^\mu\\ &\times \sum_{u=0}^{\infty} (-1)^u t^{a_u}\left(\frac{x+1}{\xi-x}\right)^{\beta_u} \tau^{|\alpha_u|+\beta_u}\left(\frac{1-2h-R}{4h}\right)^u\\ &+ \frac{1}{\xi-1}\left(\frac{\xi-x}{\xi-1}\right)^\mu \frac{z+1}{2R}\left(\frac{z-1}{z+1}\frac{1+2h-R}{1-2h-R}\right)^\mu\\ &\times \sum_{v=1}^{\infty} (-1)^v t^{a_v}\left(\frac{x+1}{\xi-x}\right)^{\beta_v} \tau^{|\alpha_v|+\beta_v}\left(\frac{1-2h-R}{4h}\right)^v \Bigg) d\xi.\end{aligned} \tag{60}$$

Recall (41), (43), (45), (42), (44), and (46) (see Table IV), then according to α_u and α_v properties, we need to break these

series on two parts each; one finite and one infinite series. Define these series as follows:

$$\sum_{u=0}^{\infty}(\cdot)^u = \sum_{u=0}^{\left\lceil\frac{\sigma-p_1+p_2}{q}\right\rceil-1}(\cdot)^u + \sum_{u=\max\left(\left\lceil\frac{\sigma-p_1+p_2}{q}\right\rceil,0\right)}^{\infty}(\cdot)^u,$$

$$\sum_{v=1}^{\infty}(\cdot)^u = \sum_{v=1}^{-\left\lceil\frac{\sigma-p_1+p_2}{q}\right\rceil}(\cdot)^v + \sum_{v=\max\left(-\left\lceil\frac{\sigma-p_1+p_2}{q}\right\rceil+1,1\right)}^{\infty}(\cdot)^v.$$

By $\tilde{\sigma}$ denote the shifted charge; then by $\tilde{n}$ denote the quotient of $\tilde{\sigma}$ and q

$$\tilde{\sigma} = \sigma - p_1 + p_2,$$
$$\tilde{n} = \left\lceil\frac{\tilde{\sigma}}{q}\right\rceil.$$

Consider (60) as four sums of integrals along the contour γ. Each of these integrals includes one of the series above. Let J_1, J_2, J_3, and J_4 denote these integrals.

$$J_1 = \frac{1}{2\pi i}\oint_\gamma \Psi_1(\xi)d\xi, \qquad J_2 = \frac{1}{2\pi i}\oint_\gamma \Psi_2(\xi)d\xi,$$
$$J_3 = -\frac{1}{2\pi i}\oint_\gamma \Psi_3(\xi)d\xi, \qquad J_4 = -\frac{1}{2\pi i}\oint_\gamma \Psi_4(\xi)d\xi,$$

where by Ψ_1, Ψ_2, Ψ_3, and Ψ_4 we denote the integrands of J_1, J_2, J_3, and J_4.

$$\Psi_1(\xi) = \frac{\tau^{\tilde{\sigma}}}{\xi - x}\left(\frac{\xi - x}{\xi - 1}\right)^\mu \frac{z+1}{2R}\left(\frac{1-2h-R}{1+2h-R}\right)^\mu \sum_{u=0}^{\tilde{n}-1} T_1^u,$$

$$\Psi_2(\xi) = \frac{\tau^{-\tilde{\sigma}}}{(\xi - x)(\xi+1)^\mu}\frac{z+1}{2R}\left(\frac{1-2h-R}{1+2h-R}\right)^\mu \times \sum_{u=\max(\tilde{n},0)}^{\infty} T_2^u, \tag{61}$$

$$\Psi_3(\xi) = \frac{\tau^{-\tilde{\sigma}}}{(\xi - x)(\xi+1)^\mu}\frac{z+1}{2R}\left(\frac{z-1}{z+1}\frac{1+2h-R}{1-2h-R}\right)^\mu \times \sum_{v=1}^{-\tilde{n}} T_1^v,$$

$$\Psi_4(\xi) = \frac{\tau^{\tilde{\sigma}}}{\xi - x}\left(\frac{\xi - x}{\xi - 1}\right)^\mu \frac{z+1}{2R}\left(\frac{z-1}{z+1}\frac{1+2h-R}{1-2h-R}\right)^\mu \times \sum_{v=\max(-\tilde{n}+1,1)}^{\infty} T_2^v, \tag{62}$$

where

$$T_1 = -\left(t/\tau\right)^q \frac{1-2h-R}{4h}, \tag{63}$$

$$T_2 = -\tau^q \frac{1-2h-R}{4h}; \tag{64}$$

then (60) becomes

$$C^\sigma(t) = \frac{t^{\lfloor -\sigma/2\rfloor + p_1}}{(1-t)^{1-\mu}}\left(J_1 + J_4\right) + \frac{2^\mu t^{\lfloor \sigma/2\rfloor + p_2}}{1-t}\left(J_2 + J_3\right). \tag{65}$$

Note that for

$$|T_2| < 1,$$

we have the convergent geometric series in (61) and (62). Therefore we can rewrite these integrands in closed form. Then we substitute (53) for h; then, in turn, $\frac{1+s}{1-s}$ for z, and (40) for s. Now, for the sake of simplicity and for eliminating, where possible, the imaginary parts of the denominators, we perform straightforward calculations and obtain

$$\Psi_1(\xi) = \frac{\tau^{\tilde{\sigma}}}{2\widetilde{R}}\left(b_d(\xi+1)\right)^{\mu}\frac{H_1 + (1-\mu\tau^q)\,\widetilde{R}}{Q} \times \left(1 - T_1^{\max(\tilde{n},0)}\right), \tag{66}$$

$$\Psi_2(\xi) = \frac{\tau^{-\tilde{\sigma}}}{2\widetilde{R}}\left(\frac{b_d(\xi-1)}{\xi-x}\right)^{\mu}\frac{H_2 + (1-\mu\,(t/\tau)^q)\,\widetilde{R}}{Q} \times T_2^{\max(\tilde{n},0)}, \tag{67}$$

$$\Psi_3(\xi) = \frac{\tau^{-\tilde{\sigma}}}{2\widetilde{R}}\left(\frac{b_d(\xi-1)}{\xi-x}\right)^{\mu}\frac{H_2 - (1-\mu\,(t/\tau)^q)\,\widetilde{R}}{Q} \times \left(1 - T_1^{\max(-\tilde{n},0)}\right), \tag{68}$$

$$\Psi_4(\xi) = \frac{\tau^{\tilde{\sigma}}}{2\widetilde{R}}\left(b_d(\xi+1)\right)^{\mu}\frac{H_1 - (1-\mu\tau^q)\,\widetilde{R}}{Q} \times T_2^{\max(-\tilde{n},0)}, \tag{69}$$

where

$$H_1 = \mu P_1 + \tau^{\mu q} P_3,$$
$$H_2 = \mu\,(t/\tau)^q\,P_2 + P_4;$$

b_k is then defined like (38) as follows:

$$b_k = \frac{t^{k+2}}{2(1-t)}.$$

Now (63) and (64) become

$$T_1 = -\,(t/\tau)^q\,\frac{P_0 - \widetilde{R}}{2b_k\,(\xi^2-1)}, \tag{70}$$

$$T_2 = -\tau^q\frac{P_0 - \widetilde{R}}{2b_k(\xi^2-1)}. \tag{71}$$

$P_0, \ldots, P_4$ are polynomials of degree 2 in the complex variable ξ.

$$P_0 = \xi - x - (b_d + b_k)\left(\xi^2 - 1\right),$$
$$P_1 = \xi - x - (b_d - b_k)\left(\xi^2 - 1\right),$$
$$P_2 = \xi - x - (b_k - b_d)\left(\xi^2 - 1\right),$$
$$P_3 = \xi - x - \left(b_d\left(1 - 2\tau^{(1-\mu)q}\right) + b_k\right)\left(\xi^2 - 1\right),$$
$$P_4 = \xi - x - \left(b_d + b_k\left(1 - 2\tau^{(\mu-1)q}\right)\right)\left(\xi^2 - 1\right).$$

Denominator functions Q, and $\widetilde{R}$ are defined as

$$Q = \xi - x - \left(b_d\left(1-\tau^q\right) + b_k\left(1-\tau^{-q}\right)\right)\left(\xi^2 - 1\right), \tag{72}$$

$$\widetilde{R} = \pm\sqrt{P_0^2 - 4b_d b_k\left(\xi^2-1\right)^2}. \tag{73}$$

The functions (72) and (73) define singularities of integrands (66), (67), (68), and (69). Indeed, $\widetilde{R}$ have 4 roots; these all roots are real:

$$\xi_{1,2} = \frac{1 \mp \sqrt{1 - 4\left(\sqrt{b_d}+\sqrt{b_k}\right)^2 x + 4\left(\sqrt{b_d}+\sqrt{b_k}\right)^4}}{2\left(\sqrt{b_d}+\sqrt{b_k}\right)^2},$$

$$\xi_{3,4} = \frac{1 \mp \sqrt{1 - 4\left(\sqrt{b_d}-\sqrt{b_k}\right)^2 x + 4\left(\sqrt{b_d}-\sqrt{b_k}\right)^4}}{2\left(\sqrt{b_d}-\sqrt{b_k}\right)^2}.$$

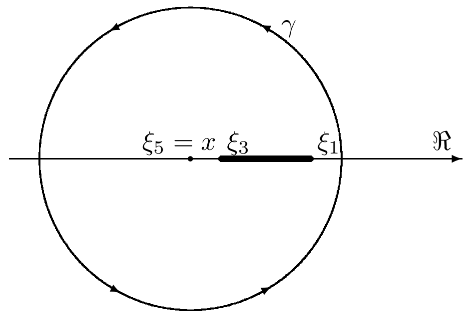

Fig. 1. Contour γ which is the circle (or any Jordan curve) about x

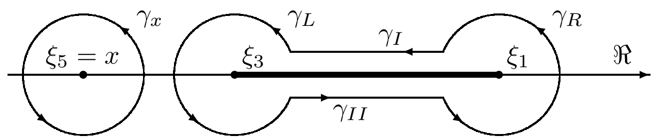

Fig. 2. Dumbbell-shaped contour about the branch cut on $[\xi_3, \xi_1]$ and a circle contour about ξ_5.

Note, that (73) becomes imaginary on $[\xi_3, \xi_1]$ and $[\xi_2, \xi_4]$; therefore (66), …, (69) have branch cuts on these segments. Obviously, that $Q = 0$ if $\xi = x$. Therefore, there might be a singularity at point $\xi_5 = x$ for integrands (66), …, (69). It may be simply proven that ξ_1, ξ_3, and ξ_5 lie inside the contour γ as shown on Fig. 1. Thick line inside the contour denotes the branch cut on $[\xi_3, \xi_1]$.

Instead of the contour γ, we can consider two contours; the first dumbbell-shaped contour whose “balls” contain, respectively, ξ_3 and ξ_1 and the second contour γ_x as shown on Fig. 2. In this case we can write J_1, say, by the following

$$\begin{aligned} J_1 = \frac{1}{2\pi i} \Bigg(& \oint_{\gamma_L} \Psi_1(\xi) d\xi + \oint_{\gamma_I} \Psi_1(\xi) d\xi \\ & + \oint_{\gamma_R} \Psi_1(\xi) d\xi + \oint_{\gamma_{II}} \Psi_1(\xi) d\xi + \oint_{\gamma_x} \Psi_1(\xi) d\xi \Bigg). \end{aligned} \tag{74}$$

We can show that the integrals along the two circles γ_L and γ_L vanish in the limit as the radius ϵ of these circles tends to 0. Indeed, we can rewrite (73) as

$$\widetilde{R} = (b_d - b_k)\sqrt{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)}.$$

Therefore, for arbitrarily small ϵ, we have

$$\begin{aligned} \widetilde{R}\Big|_{|\xi - \xi_1| \le \epsilon} &= (b_d - b_k)\sqrt{(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)} \\ & \quad \times \sqrt{(\xi - \xi_1)}, \end{aligned} \tag{75}$$

$$\begin{aligned} \widetilde{R}\Big|_{|\xi - \xi_3| \le \epsilon} &= \pm(b_d - b_k)\sqrt{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_4)} \\ & \quad \times \sqrt{(\xi - \xi_3)}. \end{aligned} \tag{76}$$

Now, we estimate integrals $\oint_{\gamma_L} \Psi_1(\xi) d\xi$ and $\oint_{\gamma_R} \Psi_1(\xi) d\xi$ as follows. First, substituting (75) and (76) for $\widetilde{R}$ in (66), we see that

$$\begin{aligned} & \frac{\tau^{\tilde{\sigma}}}{2(b_d - b_k)\sqrt{(\xi - \xi_2)(\xi - \xi_3)(\xi - \xi_4)}} (b_d(\xi + 1))^{\mu} \\ & \qquad \times \frac{H_1 + (1 - \mu\tau^q)\,\widetilde{R}}{Q} \left(1 - T_1^{\max(\tilde{n},0)}\right) \le M \end{aligned}$$

are bounded above at the point ξ_3 and it’s nearest neighborhood as well as

$$\begin{aligned} & \frac{\tau^{\tilde{\sigma}}}{2(b_d - b_k)\sqrt{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_4)}} (b_d(\xi + 1))^{\mu} \\ & \qquad \times \frac{H_1 + (1 - \mu\tau^q)\,\widetilde{R}}{Q} \left(1 - T_1^{\max(\tilde{n},0)}\right) \le M \end{aligned}$$

are bounded above at the point ξ_1 and it's nearest neighborhood. Then we show that

$$\oint_{\gamma_R} \Psi_1(\xi)d\xi \le M \oint_{\gamma_R} \frac{d\xi}{\sqrt{\xi-\xi_1}},$$
$$\oint_{\gamma_L} \Psi_1(\xi)d\xi \le M \oint_{\gamma_L} \frac{d\xi}{\sqrt{\xi-\xi_3}}.$$

Using parameterization of the contours γ_R by $\xi(\vartheta)=\xi_1+\epsilon e^{i\vartheta}$, $d\xi = i\epsilon e^{i\vartheta}d\vartheta$ with $\vartheta \in [-\pi,\pi]$ and γ_L by $\xi(\vartheta)=\xi_3+\epsilon e^{i\vartheta}$, $d\xi = i\epsilon e^{i\vartheta}d\vartheta$ with $\vartheta\in[0,2\pi]$, we have

$$\oint_{\gamma_R} \frac{d\xi}{\sqrt{\xi-\xi_1}} = 4i\sqrt{\epsilon}$$
$$\oint_{\gamma_L} \frac{d\xi}{\sqrt{\xi-\xi_3}} = -4\sqrt{\epsilon}$$

and we can see that $\oint_{\gamma_R} \Psi_1(\xi)d\xi$ and $\oint_{\gamma_L} \Psi_1(\xi)d\xi$ vanish as ϵ tends to zero.

Hence, there remain two integrals $\oint_{\gamma_I} \Psi_1(\xi)d\xi$ and $\oint_{\gamma_{II}} \Psi_1(\xi)d\xi$ along the line segments just above and just below the cut, and one integral $\oint_{\gamma_x} \Psi_1(\xi)d\xi$ (see Fig. 2).

Along the upper and along the lower segments, we can rewrite

$$\begin{aligned}
&\oint_{\gamma_I} \Psi_1(\xi)d\xi + \oint_{\gamma_{II}} \Psi_1(\xi)d\xi \\
&\quad = \int_{\xi_3+\epsilon+i0}^{\xi_1-\epsilon+i0} \Psi_1(\xi)d\xi + \int_{\xi_1-\epsilon-i0}^{\xi_3+\epsilon-i0} \Psi_1(\xi)d\xi \\
&\quad = i\int_{\xi_3}^{\xi_1} \Im\left(\Psi_1(\xi)\right)d\xi - i\int_{\xi_1}^{\xi_3} \Im\left(\Psi_1(\xi)\right)d\xi, \quad \text{as } \epsilon\to 0.
\end{aligned}$$

This directly follows from the fact that $\widetilde{R}$ is the only function which becomes imaginary into these limits of integration. Obviously, that (74) becomes

$$J_1 = \frac{1}{\pi}\int_{\xi_3}^{\xi_1} \Im\left(\Psi_1(\xi)\right)d\xi + \operatorname*{Res}_{\xi=x}(\Psi_1(\xi)). \tag{77}$$

In the same way, we get $J_2,\ldots,J_4$ in the form identical with (77).

It is not hard to show that

$$\frac{P_0-\widetilde{R}}{2b_k(\xi^2-1)} = t^{-q/2}e^{i\varphi} \tag{78}$$

between the limits of integration $[\xi_3,\xi_1]$, where we define φ as

$$\begin{aligned}
\varphi &= -i\ln\left(\frac{P_0-\widetilde{R}}{2\sqrt{b_d b_k}(\xi^2-1)}\right) \\
&= -\arccos\left(\frac{P_0}{2\sqrt{b_d b_k}(\xi^2-1)}\right).
\end{aligned} \tag{79}$$

Equation (79) directly follows from the fact that $\widetilde{R}$ becomes imaginary on segment $[\xi_3,\xi_1]$.

Now, recall (70) and (71); then consider power functions $T_1^{\hat{n}}$ and $T_2^{\hat{n}}$. Taken into account (78), we can rewrite these functions as

$$T_1^{\hat{n}} = (-1)^{\hat{n}}\left(\frac{\sqrt{t}}{\tau}\right)^{\hat{n}q}\left(\cos\hat{n}\varphi + i\sin\hat{n}\varphi\right),$$
$$T_2^{\hat{n}} = (-1)^{\hat{n}}\left(\frac{\sqrt{t}}{\tau}\right)^{-\hat{n}q}\left(\cos\hat{n}\varphi + i\sin\hat{n}\varphi\right),$$

where $\hat{n}$ denotes either $\max(\tilde{n},0)$ or $\max(-\tilde{n},0)$.

Since we need only imaginary parts of (66), (67), (68), and (69), then we can rewrite the sums J_1+J_4 and J_2+J_3 in (65) as follows:

$$\begin{aligned}
J_1+J_4 = \frac{(-1)^{\tilde{n}}}{2\pi}&\int_{\xi_3}^{\xi_1} \frac{\tau^{\tilde{\sigma}}}{Q}\left(b_d(\xi+1)\right)^{\mu} \\
&\times\left(\frac{\sqrt{t}}{\tau}\right)^{\tilde{n}q}\left(\frac{H_1}{|\widetilde{R}|}\cos\left(\tilde{n}\varphi\right) - (1-\mu\tau^q)\sin\left(\tilde{n}\varphi\right)\right)d\xi \\
&\qquad + \operatorname*{Res}_{\xi=x}(\Psi_1(\xi)) - \operatorname*{Res}_{\xi=x}(\Psi_4(\xi))
\end{aligned} \tag{80}$$

and

$$J_2+J_3=-\frac{(-1)^{\tilde{n}}}{2\pi}\int_{\xi_3}^{\xi_1}\frac{\tau^{-\tilde{\sigma}}}{Q}\left(\frac{b_d(\xi-1)}{\xi-x}\right)^{\mu}$$
$$\times\left(\frac{\sqrt{t}}{\tau}\right)^{-\tilde{n}q}\left(\frac{H_2}{|\widetilde{R}|}\cos(\tilde{n}\varphi)-(1-\mu\,(t/\tau)^q)\sin(\tilde{n}\varphi)\right)d\xi$$
$$+\operatorname*{Res}_{\xi=x}(\Psi_2(\xi))-\operatorname*{Res}_{\xi=x}(\Psi_3(\xi)). \tag{81}$$

To conclude the derivation of our generating function, it remains to note that

$$\operatorname*{Res}_{\xi=x}(\Psi_1(\xi))=0,$$
$$\operatorname*{Res}_{\xi=x}(\Psi_2(\xi))=\frac{\mu t^{q\max(\tilde{n},0)-\tilde{\sigma}}(t-1)}{2\,(1-t^q)}, \tag{82}$$
$$\operatorname*{Res}_{\xi=x}(\Psi_3(\xi))=0,$$
$$\operatorname*{Res}_{\xi=x}(\Psi_4(\xi))=0.$$

The proof of (82) is straightforward but tedious and omitted here.

Degree of radicand of $\widetilde{R}$ and properties of its roots define (80) and (81) as elliptic. Therefore, generating function (31) does not have a closed form.

Thus, we just proved the following theorem.

Theorem 1. *A recursion relation* (6) *does not have a closed form.*

Note that from the residue theorem it follows that the limits ξ_3 and ξ_1 may be replaced by ξ_2 and ξ_4 respectively with changing signs of (80) and (81).

D. Generating Function of Sequence $C_n^{-n}, C_n^{-n+2}, \ldots, C_n^n$

Consider a finite power series

$$C_n(y)=\sum_{\substack{\sigma=-n\\ \sigma\ \text{even}}}^{n}C_n^{\sigma}y^{\sigma/2},\quad\text{or}\quad C_n(y)=\sum_{\substack{\sigma=-n\\ \sigma\ \text{odd}}}^{n}C_n^{\sigma}y^{(\sigma-1)/2},$$

where $y\in\mathbb{R}$.

Here we can also consider (33) instead of C_n^{σ}. The calculations that let us got (34) yield now

$$C_n(y)=\sum_{\substack{m=\rho\\ m\ \text{even}\\ \text{or}\\ m\ \text{odd}}}^{2n+\rho}\left(\sum_{u=0}^{\frac{m-\rho}{2}-\mu}\sum_{v=0}^{\frac{m-\rho}{2}}(-1)^{u+v}\right.$$
$$\left.\times\binom{\frac{m-\rho}{2}-\mu}{u}\binom{\frac{m-\rho}{2}}{v}g(y)\right), \tag{83}$$

where by $g(y)$ we denote the inner sum as follows:

$$g(y)=\sum_{\substack{\sigma=m\\ \sigma\ \text{even}\\ \text{or}\\ \sigma\ \text{odd}}}^{2n+\rho}\binom{\frac{\sigma-\rho}{2}-p_1-\frac{m-\rho}{2}d-uq-\mu}{\frac{m-\rho}{2}-\mu}$$
$$\times\binom{n-\frac{\sigma-\rho}{2}-p_2-\frac{m-\rho}{2}d-vq}{\frac{m-\rho}{2}}y^{\frac{-n+\sigma}{2}-\rho}. \tag{84}$$

Certainly, we may use the line of reasoning that has led us from (27) to (29). However, we choose another way. It is easy to observe that n and σ form one term $\frac{n+\sigma}{2}$ or $\frac{n-\sigma}{2}$ (see (33)). So we can repeat the steps between (33) and (39) and obtain

$$g(y)=(-1)^{\alpha}y^{-\frac{n+\rho}{2}+a+\frac{m-\rho}{2}d}(x-1)^{\alpha+m-\rho+1}(x+1)^{1-\mu}$$
$$\times\frac{1}{2\pi i}\oint_{\gamma}\frac{(\xi+1)^{\mu}}{(\xi^2-1)^{\frac{m-\rho}{2}+1}(\xi-x)^{\alpha+1}}d\xi, \tag{85}$$

where we redefine α and a as

$$\begin{aligned}\alpha &= n-(m-\rho)(d+1)-p_1-p_2-(u+v)q,\\ a &= p_1+uq.\end{aligned}$$

Note that in this case we do not need the hypergeometric transformation that we have performed above. Indeed, we already have finite series by definition (84). If we replace $\frac{m-\rho}{2}$ by $\tilde{m}$, we get

$$\sum_{\substack{m=\rho\\ m \text{ even}\\ \text{or}\\ m \text{ odd}}}^{2n+\rho}(\cdot)=\sum_{\tilde{m}=0}^{n}(\cdot).$$

After interchanging the order of summation and integration in (83), we see that each of two inner sums can be identified with the binomial theorem.

$$\begin{aligned}C_n(y) &= (-1)^{\alpha} y^{-\frac{n+\rho}{2}}(x-1)^{n-p_2+1}(x+1)^{1-p_1-\mu}\\ &\times \frac{1}{2\pi i}\oint_{\gamma}\frac{(\xi+1)^{\mu}}{(\xi^2-1)(\xi-x)^{n-p_1-p_2+1}}\\ &\times \left(1-\mu+\sum_{\tilde{m}=1}^{n}\left(\frac{(\xi-x)^{2(d+1)}}{(x^2-1)^d(\xi^2-1)^{n-p_1-p_2+1}}\right)^{\tilde{m}}\right.\\ &\left.\times\left(1-\left(\frac{x-\xi}{x+1}\right)^q\right)^{\tilde{m}-\mu}\left(1-\left(\frac{x-\xi}{x-1}\right)^q\right)^{\tilde{m}}\right)d\xi. \qquad (86)\end{aligned}$$

Now, only a finite geometric series remains in (86). Redefine h as a term of this series

$$\begin{aligned}h = &\frac{(\xi-x)^{2(d+1)}}{(x^2-1)^d(\xi^2-1)^{n-p_1-p_2+1}}\\ &\times\left(1-\left(\frac{x-\xi}{x+1}\right)^q\right)\left(1-\left(\frac{x-\xi}{x-1}\right)^q\right).\end{aligned}$$

Substituting $\frac{h-h^{n+1}}{1-h}$ for $\sum_{\tilde{m}=1}^{n}h^{\tilde{m}}$ in (86), we get

$$\begin{aligned}C_n(y) &= (-1)^{\alpha} y^{-\frac{n+\rho}{2}}(x-1)^{n-p_2+1}(x+1)^{1-p_1-\mu}\\ &\times\left(\frac{1-\mu}{2\pi i}\oint_{\gamma}\frac{(\xi+1)^{\mu}}{(\xi^2-1)(\xi-x)^{n-p_1-p_2+1}}d\xi\right.\\ &\left.+\frac{1}{2\pi i}\oint_{\gamma}\Psi(\xi)d\xi-\frac{1}{2\pi i}\oint_{\gamma}\Psi(\xi)h^n d\xi\right), \qquad (87)\end{aligned}$$

where by $\Psi(\xi)$ we denote the integrand as follows:

$$\begin{aligned}\Psi(\xi) &= \frac{(\xi+1)^{\mu}}{(\xi^2-1)(\xi-x)^{n-p_1-p_2+1}}\\ &\times\left(1-\left(\frac{x-\xi}{x+1}\right)^q\right)^{-\mu}\frac{h}{1-h}\\ &= \left(x^2-1\right)^{k+1}(x+1)^{\mu q}\\ &\times\frac{(\xi+1)^{\mu}h}{(\xi-x)^{n-p_1-p_2+1}\psi\left((x+1)^q-(x-\xi)^q\right)^{\mu}},\end{aligned}$$

where

$$\begin{aligned}\psi(\xi) &= \left(x^2-1\right)^{k+1}\left(\xi^2-1\right)\\ &-(x-\xi)^{2(d+1)}\left(x^2-1\right)^q\\ &+(x-\xi)^{(d+1)+(k+2)}\left((x-1)^q+(x+1)^q\right)\\ &-(x-\xi)^{2(k+2)}\end{aligned}$$

such that $1-h=\frac{\psi(\xi)}{(x^2-1)^{k+1}(\xi^2-1)}$. Now if we recall that the contour γ was initially considered as a small circle surrounding the point x, then, using the residue theorem, we get

$$\begin{aligned}\frac{1}{2\pi i}\oint_\gamma \Psi(\xi)d\xi &= \left(x^2-1\right)^{k+1}(x+1)^{\mu q}\\ &\times \sum_{j=1}^{2(k+2)} \frac{\lim\limits_{\xi\to\xi_j} h}{(\xi_j-x)^{n-p_1-p_2+1}\prod\limits_{\substack{m=1,\\ m\neq j}}^{2(k+2)}(\xi_j-\xi_m)}\\ &\times \left(\begin{cases}\dfrac{\xi_j+1}{(x+1)^q-(x-\xi_j)^q}, & \xi_j\neq -1,\\ \dfrac{1}{q(x-\xi_j)^{q-1}}, & \text{otherwise},\end{cases}\right)^{\mu},\end{aligned} \tag{88}$$

where

$$\lim_{\xi\to\xi_j} h = \begin{cases}\dfrac{q\,(x-\xi_j)^{d+1}\left((x-\xi_j)^q-(x+\xi_j)^q\right)}{2\xi_j\,(x+\xi_j)^{k+1}}, & |\xi_j|=1,\\ 1, & \text{else},\end{cases}$$

and $\xi_1,\ \xi_2,\ \ldots,\ \xi_{2(k+2)}$ are roots of $\psi(\xi)$.

It can easily be checked that

$$\frac{1}{2\pi i}\oint_\gamma \Psi(\xi)h^n d\xi = 0. \tag{89}$$

We might prove it using the contour integration methods. But more simply is to observe that $C_n^\sigma=0$ whenever $\sigma>n$.

It remains to find the first integral in (87). Obviously,

$$\begin{aligned}&\frac{1-\mu}{2\pi i}\oint_\gamma \frac{(\xi+1)^\mu}{(\xi^2-1)(\xi-x)^{n-p_1-p_2+1}}d\xi\\ &=(1-\mu)\left(\frac{1}{2(-1-x)^{n-p_1-p_2+1}}-\frac{1}{2(1-x)^{n-p_1-p_2+1}}\right).\end{aligned} \tag{90}$$

Note that these integrals become equal to 0 whenever $n-p_1-p_2+1\le 0$.

Substituting (90), (88), and (89) for the first, for the second, and for the third integrals in (87), we obtain the final result.

V. Algorithms for Encoding and Decoding Constant-Weight and Constant-Charge Binary Run-Length Limited Sequences

Recall that $\hat{\mathscr{S}}$ denotes a set of constant-weight or constant-charge binary run-length constrained sequences $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ of length n. Let the set $\hat{\mathscr{S}}$ be ordered lexicographically. From [3] it follows that the lexicographic index of $\boldsymbol{x}\in\hat{\mathscr{S}}$ is given by

$$N(\boldsymbol{x})=\sum_{j=1}^{n} x_j W(\boldsymbol{p}), \tag{91}$$

where $W(\boldsymbol{p})$ denotes the number of sequences in $\hat{\mathscr{S}}$ with given prefix $\boldsymbol{p}=(x_1,x_2,\ldots,x_{j-1},0)$.

The decoding algorithm, for given sequence $\boldsymbol{x}$, find its lexicographic index N, $0\le N<|\hat{\mathscr{S}}|$. This is done by successive approximation method [3], [15] using (91) and $W(\boldsymbol{p})$ as the number of sequences in $\hat{\mathscr{S}}$ with given prefix $\boldsymbol{p}=(x_1,x_2,\ldots,x_{j-1},0)$.

By $a_j(\boldsymbol{p})$ denote the number of trailing zeros of the prefix $\boldsymbol{p}$. By $\nu_j(\boldsymbol{p})=\nu_{j-1}=\sum_{i=1}^{j}x_i$ and by $\sigma_j(\boldsymbol{p})=\sum_{i=1}^{j}(-1)^{\nu_i}$ denote the weight and the charge of this prefix. Since $\boldsymbol{p}$ is the prefix of $\boldsymbol{x}$, then subsequence $\tilde{\boldsymbol{x}}=(x_j,x_{j+1},\ldots,x_n)$ is the rest of $\boldsymbol{x}$, and l_j is the leading run of zeros in this subsequence. We define l_j as the complement of $a_j(\boldsymbol{p})$ in l (for leading run of zeros) or in k (for another run of zeros) as follows:

$$l_j=\begin{cases} l-a_j(\boldsymbol{p}), & \nu_{j-1}=0,\\ k-a_j(\boldsymbol{p}), & \text{otherwise}.\end{cases}$$

Similarly, by r_j we denote the trailing run of zeros in the subsequence $\tilde{\boldsymbol{x}}$. If this subsequence consists of zeros, this may mean either the leading run of zeros (when $\nu=0$), or the trailing run of zeros. Therefore, we define r_j as the complement of $a_j(\boldsymbol{p})$

either in $\min(l, r)$, or in r. In the case of nonzero $\tilde{\boldsymbol{x}}$, we define r_j as r, i.e.,

$$r_j = \begin{cases} \min(l, r) - a_j(\boldsymbol{p}), & \nu = \nu_{j-1} = 0, \\ r - a_j(\boldsymbol{p}), & \nu = \nu_{j-1} \neq 0, \\ r, & \nu \neq \nu_{j-1}, \end{cases}$$

or

$$r_j = \begin{cases} \min(l, r) - a_j(\boldsymbol{p}), & \sigma_{n-j}(\boldsymbol{p}) = n - j \text{ and } \nu_{j-1} = 0, \\ r - a_j(\boldsymbol{p}), & \sigma_{n-j}(\boldsymbol{p}) = n - j \text{ and } \nu_{j-1} \neq 0, \\ r, & \sigma_{n-j}(\boldsymbol{p}) \neq n - j, \end{cases}$$

where

$$\sigma_{n-j}(\boldsymbol{p}) = (-1)^{\nu_{j-1}}(\sigma - \sigma_{j-1}) - 1.$$

Then we can compute the number of constant-weight sequences $W_\nu(\boldsymbol{p})$ as

$$W_\nu(\boldsymbol{p}) = \begin{cases} \hat{A}_{n-j}^{\nu-\nu_j}(d, k, l_j, r_j), & l_j \geq 0 \text{ and } r_j \geq 0, \\ 0, & \text{otherwise} \end{cases}$$

and the number of constant-charge sequences $W_\sigma(\boldsymbol{p})$ as

$$W_\sigma(\boldsymbol{p}) = \begin{cases} \hat{C}_{n-j}^{\sigma_{n-j}(\boldsymbol{p})}(d, k, l_j, r_j), & l_j \geq 0 \text{ and } r_j \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{\gamma} = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ be a binary vector; components of this vector γ_j, $0 \leq j \leq n$ indicate that $\boldsymbol{x}$ does (if $\gamma_j = 1$) or does not (if $\gamma_j = 0$) have the weight $\nu = j$ or the charge $\sigma = 2j - n$.

Hence, the number $W(\boldsymbol{p})$ of sequences with given prefix $\boldsymbol{p}$ be

$$W(\boldsymbol{p}) = \begin{cases} \displaystyle\sum_{\nu=0}^{n} \gamma_\nu W_\nu(\boldsymbol{p}), & \text{constant-weight case,} \\ \displaystyle\sum_{\substack{\sigma=-n \\ \sigma \text{ even} \\ \text{or} \\ \sigma \text{ odd}}}^{n} \gamma_{(\sigma+n)/2} W_\sigma(\boldsymbol{p}), & \text{constant-charge case.} \end{cases}$$

We introduce the next variables: a, w, c, which correspond to $a_j(\boldsymbol{p})$, ν_{j-1}, σ_{j-1}.

$N := 0$; $a := 1$; $w := 0$; $c := 0$;
for $j := 1$ **to** n **do**
  Get $W(\boldsymbol{p})$;
  if $x_j = 1$ **then**
    $N := N + W(\boldsymbol{p})$;
    $a := 1$; $w := w + 1$;
  else
    $a := a + 1$;
  end if . . . else
  $c := c + (-1)^w$;
end for.

The encoding (inverse) algorithm, for given lexicographic index N, $0 \leq N < |\hat{\mathscr{S}}|$, find the corresponding $\boldsymbol{x}$.

$a := 1$; $w := 0$; $c := 0$;
for $j := 1$ **to** n **do**
  Get $W(\boldsymbol{p})$;
  if $N \geq W(\boldsymbol{p})$ **then**
    $N := N - W(\boldsymbol{p})$;
    $x_j := 1$; $a := 1$; $w := w + 1$;
  else
    $x_j := 0$; $a := a + 1$;
  end if . . . else
  $c := c + (-1)^w$;
end for.

TABLE V

AN EXAMPLE OF SINGLE PEAK SHIFT

N	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	ν	σ	
0	0	1	0	0	0	0	**1**	0	2		right
	1	-1	-1	-1	-1	-1	1	1		-2	shift
1	0	1	0	0	0	**1**	0	0	2		true
	1	-1	-1	-1	-1	1	1	1		0	position
2	0	1	0	0	**1**	0	0	0	2		left
	1	-1	-1	-1	1	1	1	1		2	shift

VI. FURTHER REMARKS

A.

As it was shown in Section II, the recursion relation (6) has an implicit mutual nature. We can cite an example of practically identical coding scheme [16] in which we obtained mutual recursions in explicit form. Alternating runs in [16] were presented by series of zeros or ones with independent constraints. It seems now; there was a heavy construction without significant preference of present constant-charge code.

B.

Channels may cause peak shifts in dk-constrained sequences [17]. It is considered as a more frequent error. We provide an example (see Table V) which shows that the charge distribution seems rather suitable for the peak shift analysis then the weight distribution. On the other hand, the weight distribution remains useful for erasure and insertion control. In more detail, consider the integer (or composition) representation of RLL sequences [18]. This representation is used in error detection and correction technique including the peak shifts correction [19], [20]. In such case an RLL sequence is parsed uniquely into a concatenation of phrases, each phrase beginning with one. By ϕ_j denote the length of jth phrase. We can state a simple relation between ϕ_j and charge σ_j as follows:

$$\phi_j = j - i = \frac{\sigma_j - \sigma_i}{(-1)^{\nu_j}},$$

where i and j are positions of consecutive ones such that $i < j$. Moreover, from Section II and Section V it follows that Cover's enumerative scheme implies counting of ϕ_j.

C.

Since run-length constraints bound the weight distribution, we have $\nu \in [\nu_{\min}, \nu_{\max}]$, where $0 \le \nu_{\min} \le \nu_{\max} \le n$. In [6], Ytrehus obtained

$$\nu_{\min} = \begin{cases} 0, & n \le \min(l,r), \\ 1, & \min(l,r) < n \le l+r+1, \\ \left\lceil \dfrac{n-l-r-1}{k+1} \right\rceil + 1, & l+r+1 < n \end{cases}$$

and

$$\nu_{\max} = \left\lceil \frac{n}{d+1} \right\rceil$$

for $dklr$ sequences. Similarly, for the charge distribution of dk sequences we can write

$$|\sigma|_{\max} = \begin{cases} n - 2m(d+1), & 0 \le n - m(k+d+2) \\ & \quad \le k+1, \\ 2(m+1) & \\ \quad \times (k+1) - n, & k+1 \le n - m(k+d+2) \\ & \quad \le k+d+2, \end{cases}$$

where

$$m = \left\lfloor \frac{n}{k+d+2} \right\rfloor .$$

D.

The charge of the prefix σ_j (see Section II and Section V) is called the running digital sum (RDS), see [21]. The RDS has a finite range of values. For the RLL sequences with zero accumulated charge ($\sigma_n = 0$), we can see that absolute value of RDS does not exceed $\lfloor |\sigma|_{\max}/2 \rfloor + 1$.

We can impose restrictions on the range of RDS values. Denote by ϑ_1 and ϑ_2 the lower and the upper bounds of this constrained range. The range of RDS bounded by ϑ_1 and ϑ_2 is said to be the digital sum variation (DSV), see [22], [23]. After the authors [23], DSV constrained RLL sequences are called DCRLL sequences.

By $C_n^\sigma(d,k,r,\vartheta_1,\vartheta_2)$ denote the number of DCRLL sequences beginning with one. Also by $\hat{C}_n^\sigma(d,k,l,r,\vartheta_1,\vartheta_2)$ we denote the number of DCRLL sequences beginning with a leading run of zeros. Below under $C_n^\sigma(\vartheta_1,\vartheta_2)$ we consider $C_n^\sigma(d,k,r,\vartheta_1,\vartheta_2)$ and under $\hat{C}_n^\sigma$ we similarly consider $\hat{C}_n^\sigma(d,k,l,r,\vartheta_1,\vartheta_2)$.

As shown in [24], calculation of the number of DCRLL sequences can be performed as concatenation of two subcodes; the volume of each subcode is found recurrently in explicit mutual form.

The recurrent method for calculating the number of constant-charge RLL sequences, which we suggest in Section II, allows us simply turn to calculating the number of DCRLL sequences. Indeed, we know the charge value of the prefix $\boldsymbol{p}$ at the each level of recursion (6). Therefore we can control DSV by using the additional condition $\vartheta_{1m} \leq \sigma_m \leq \vartheta_{2m}$. For this reason, we do not account $C_m^{\sigma_m}$ for which this condition does not satisfy. Here under ϑ_{1m} and ϑ_{2m} we consider ϑ_1 and ϑ_2 after justification at the each level of recursion (6). Moreover, each level of this recursion alternates the direction of charge changing; then it is sufficient to check either $\vartheta_{1m} \leq \sigma_m$ or $\sigma_m \leq \vartheta_{2m}$ condition depending on the direction of charge changing. By substituting ϑ_1 for ϑ_2 and vice versa, when calling $C_n^\sigma(\vartheta_1,\vartheta_2)$, we keep the only condition $\vartheta_{1m} \leq \sigma_m$ and obtain the implicit mutual recursion similar to (6).

From initial conditions (5) it follows that a unique sequence of zero length and zero charge exists. In turn, from this statement it follows that we need an additional condition; this condition allows us to take into account the existence of $C_0^0(\vartheta_1,\vartheta_2)$. Indeed, the sequences, which beginning with zero, have initial charge equal to 1, the sequences, which beginning with one, have initial charge equal to -1, and the sequence of zero length, have initial charge equal to 0. We now write this triple condition using the Iverson bracket notation

$$[\text{condition}] = \begin{cases} 1, & \text{the condition is true,} \\ 0, & \text{the condition is false.} \end{cases}$$

Thus we can rewrite Proposition 2 as

Proposition 3. *The numbers $C_n^\sigma(\vartheta_1,\vartheta_2)$ and $\hat{C}_n^\sigma$ can be obtained as:*
If $-[n \neq 0] > \vartheta_2$, then

$$C_n^\sigma(\vartheta_1,\vartheta_2) = 0.$$

If $\sigma = -n$ and the sequences begin with one, then

$$C_n^\sigma(\vartheta_1,\vartheta_2) = \begin{cases} 1, & n \leq \min(r+1, -\vartheta_1), \\ 0, & \textit{otherwise}. \end{cases}$$

If $\sigma \neq -n$ and the sequences begin with one, then

$$C_n^\sigma(\vartheta_1,\vartheta_2) = \begin{cases} \displaystyle\sum_{j=d+1}^{\min(n,k+1,-\vartheta_1)} C_{n-j}^{-\sigma-j}(-\vartheta_2-j,-\vartheta_1-j), & d+1 \leq n, \\ 0, & \textit{otherwise}. \end{cases}$$

If $\vartheta_1 > [n \neq 0]$, then

$$\hat{C}_n^\sigma = 0.$$

If $\sigma = n$ and a leading series is running, then

$$\hat{C}_n^\sigma = \begin{cases} 1, & n \leq \min(l, r, \vartheta_2), \\ 0, & \textit{otherwise}. \end{cases}$$

If $\sigma \neq n$ and a leading series is running, then

$$\hat{C}_n^\sigma = C_n^\sigma(\vartheta_1,\vartheta_2) + \sum_{j=1}^{\min(n,l,\vartheta_2)} C_{n-j}^{\sigma-j}(\vartheta_1-j,\vartheta_2-j).$$

Evidently, the algorithms from Section V are suitable for encoding and decoding DCRLL sequences if $W_\sigma(\boldsymbol{p})$ will be calculated as

$$W_\sigma(\boldsymbol{p}) = \begin{cases} \hat{C}_{n-j}^{\sigma_{n-j}(\boldsymbol{p})}(d,k,l_j,r_j,\vartheta_{1j},\vartheta_{2j}), & l_j \geq 0 \text{ and } r_j \geq 0, \\ & \text{and} \\ & \vartheta_1 \leq \sigma_j(\boldsymbol{p}) \leq \vartheta_2, \\ 0, & \text{otherwise.} \end{cases}$$

where

$$\begin{aligned} \sigma_j(\boldsymbol{p}) &= \sigma_{j-1} + (-1)^{\nu_{j-1}}, \\ \vartheta_{1j} &= \left(\begin{cases} \vartheta_1, & \text{if } \nu_{j-1} \text{ is even,} \\ -\vartheta_2, & \text{otherwise;} \end{cases} \right) - (-1)^{\nu_{j-1}}\sigma_j(\boldsymbol{p}), \\ \vartheta_{2j} &= \frac{(-1)^{\nu_{j-1}}(\vartheta_1+\vartheta_2) - \vartheta_1 + \vartheta_2}{2} - (-1)^{\nu_{j-1}}\sigma_j(\boldsymbol{p}). \end{aligned}$$

E.

One can find in literature some examples of application of generating functions for RLL coding. From 48 years range of publications [25] – [26], we cite two examples. Kolesnik and Krachkovsky [12], when deriving the Gilbert-Varshamov bound, estimated the volume $V_r(\boldsymbol{x})$ of sphere in $\mathscr{S}(n)$ centered on $\boldsymbol{x}$ as

$$V_r(\boldsymbol{x}) \leq \min_{0 \leq y \leq 1} \frac{A_n(y)}{y^r A_n(0)},$$

where $A_n(y)$ may be taken from (30). Note, that practical applications of the generating functions for enumerating RLL sequences often require a rational expression. Ferreira and Lin [19] and some other authors [4], [7] obtained a generating function for enumerating dk and dkr sequences

$$\mathscr{S}(t) = \frac{t(1-t^{r+1})}{1-t-t^{d+1}+t^{k+2}}.$$

Assume $y = 1$. Then from (28) we immediately get the same.

F.

Recall (57), (58), and (59). One can identify I_u and I_v as generating functions of the Jacobi polynomials [14]. Then $\cos(\tilde{n}\varphi)$ and $\sin(\tilde{n}\varphi)$ in (80) and in (81) can be identified with Chebyshev polynomials in the variable $\frac{P_0}{2\sqrt{b_d b_k(\xi^2-1)}}$. The technique which uses the theory of orthogonal polynomials is known in charge constrained coding [27].

G.

We obtain the generating function (31) in the form of elliptic integrals (80) and (81). Further exploration may require a canonical form of these integrals. Using factorization, it is not so hard to reduce the first term of the integrands in (80) and (81) to rational functions. In such case, we shall use polynomial expansion of $\cos(\tilde{n}\varphi)$. The next transformations, which bring these integrals to canonical form, may be taken from [14].

H.

If we need a two-variable generating function for the case of constant-charge sequences

$$C(t,y) = \sum_{\substack{\sigma=c_1 \\ \sigma \text{ even} \\ \text{or} \\ \sigma \text{ odd}}}^{c_2} \sum_{\substack{n=\rho \\ n \text{ even} \\ \text{or} \\ n \text{ odd}}}^{\infty} C_n^\sigma t^{(n-\rho)/2} y^{(\sigma-\rho)/2},$$

then we can write

$$C(t,y) = \sum_{\substack{\sigma=c_1 \\ \sigma \text{ even} \\ \text{or} \\ \sigma \text{ odd}}}^{c_2} C^\sigma(t) y^{(\sigma-\rho)/2}.$$

In such case we may perform summation of (66), (67), (68), and (69) over the range $[c_1, c_2]$ of even or odd numbers. In result we do not obtain any additional poles with nonzero residues. So, there remain the contours of integration depicted on Fig 2.

VII. CONCLUSION

We have presented constant-weight and constant-charge run-length constrained binary sequences. On the base of Cover's enumerative technique, we have obtained recursion relations for calculating the numbers of these sequences. For investigation of the asymptotic behavior of these values, we have derived generating functions for enumerating such sequences. We have proved that generating function for enumerating constant-charge sequences does not be expressed in a closed form. So, we have presented the long chain of derivation steps which led us to expression for this generating function in the form of elliptic integrals. Also we have described two algorithms for enumerative encoding and decoding constant-weight and constant-charge sequences. Then, we have provided some examples of application for our results; in particular, we have extended our results on RDS constrained sequences.

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